The Existence of Exactly m-coloured Complete Subgraphs

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1. Introduction

The classical result of Ramsey [6] for colourings of infinite graphs can be stated in the following way.

Theorem 1. *Let $c \geq 1$ be a positive integer and suppose that we have a c-colouring of the edges of the complete graph whose vertex set is $\mathbb{N}$, i.e. a function*

$$\Delta : \mathbb{N}^{(2)} \to \{1, \dots, c\}.$$

Then there is an infinite complete subgraph all of whose edges have the same colour. □

This theorem has inspired very many generalizations and further questions. Much of this work concerns the existence of particular monochromatic structures, but one might also take some $m > 1$ and search for substructures where exactly m colours are used. In [2] M. Erickson formulated such a problem in the form of a proposition $P(c, m)$, for positive integers c, m:

P(c,m): *If the edges of a countable infinite complete graph K_ω are exactly c-coloured, then there exists a countable infinite complete subgraph H of K_ω whose edges are exactly m-coloured.*

Of course, a graph G is said to be *exactly x-coloured* ($x \in \mathbb{N}$) if the colouring map from $E(G)$ to $\{1, \ldots, x\}$ is surjective. As usual we denote the vertex set of a graph $V(G)$ and we denote the edge set $E(G)$.

The case $m = 1$ of $P(c, m)$ is, of course, just Ramsey's Theorem, and $P(c, m)$ trivially holds in the case $c = m$. Erickson [2] observed that a fairly straightforward application of Ramsey's Theorem also enables one to show that $P(c, m)$ is true in the case $m = 2$ (provided $c \geq 2$; of course $P(c, m)$ is false if $c < m$). Erickson [2] found counterexamples to $P(c, m)$ in many other cases and he conjectured that the only cases for which $P(c, m)$ holds are those just described. In other words, he conjectured that $P(c, m)$ is true if and only if

$$\begin{aligned} &(1) \quad m = 1, \quad \text{or} \\ &(2) \quad m = 2 \text{ and } c \geq 2, \quad \text{or} \\ &(3) \quad c = m \geq 3. \end{aligned}$$

It is not too difficult to produce various families of counterexamples to $P(c, m)$ each family sustaining the conjecture of Erickson for a significant range of values for c and m. The principal difficulty in proving the conjecture seems to be that one family of counterexamples will cover values with a certain property, and another family will cover values with some quite unrelated property. Even if we put together several such families of counterexamples, there always seem to be some parameter values which are not covered.

We are not able to give a complete solution to the conjecture. However, we do obtain counterexamples to $P(c, m)$ which are significantly different from those obtained before; with the aid of random methods we are then able to extend these counterexamples to show that for each $m \geq 3$, $P(c, m)$ is false for all sufficiently large c.

In Section 2 of this paper we will describe our main new method for constructing counterexamples. This will enable us to prove the following result.

Theorem 2. *Suppose that $c > m \geq 3$ are positive integers. Let n, p, k and q be the unique natural numbers such that*

$$c = \binom{n}{2} + 2 - p \quad \text{and} \quad 0 \leq p \leq n - 2,$$

$$m = \binom{k}{2} + 2 - q \quad \text{and} \quad 0 \leq q \leq k - 2.$$