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Independent Finite Sums for K_m-free Graphs

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Abstract. Recently, in conversation with Erdős, Hajnal asked whether or not for any triangle-free graph G on the vertex set $\mathbb{N}$, there always exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ so that whenever F and H are distinct finite nonempty subsets of $\mathbb{N}$, $\{\Sigma_{n\in F}\ x_n, \Sigma_{n\in H}\ x_n\}$ is not an edge of G (that is, $FS(\langle x_n\rangle_{n=1}^{\infty})$ is an independent set). We answer this question in the negative. We also show that if one replaces the assumption that G is triangle-free by the assertion that for some m, G contains no complete bipartite graph $K_{m,m}$, then the conclusion does hold. If for some $m \geq 3$, G contains no K_m, we show there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ so that whenever F and H are *disjoint* finite nonempty subsets of $\mathbb{N}$, $\{\Sigma_{n\in F}\ x_n, \Sigma_{n\in H}\ x_n\}$ is not an edge of G. Both of the affirmative results are in fact valid for a graph G on an arbitrary semigroup $(S, +)$.

1. Introduction.

We take $\mathbb{N}$ to be the positive integers and $\omega = \mathbb{N}\cup\{0\}$. Given a set A, we denote by $\mathcal{P}_f(A)$ the set of finite nonempty subsets of A. Given a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, we use the notation $FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\Sigma_{n\in F}\ x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. In 1972 the following theorem was proved in [5] (or see [1] or [6] for simpler proofs).

1.1 Theorem. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i$.*

It was already known at the time [4] that Theorem 1.1 is equivalent to the superficially weaker version which has $r = 2$. In 1995, Hajnal asked Erdős the following.

1.2 Question. *Let G be a graph on the vertex set $\mathbb{N}$ which contains no triangles. Must there exist a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})$ forms an independent set?*

Note that an affirmative answer to Question 1.2 would imply Theorem 1.1. To see this, let $\mathbb{N} = A_1\cup A_2$ where we may presume that $A_1\cap A_2 = \emptyset$. (We have already observed that it suffices to establish Theorem 1.1 for the case $r = 2$.) Let G be the complete

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bipartite graph on the sets A_1 and A_2. That is, $E(G) = \big\{\{x,y\} : x \in A_1 \text{ and } y \in A_2\big\}$. Then if $\langle x_n\rangle_{n=1}^\infty$ has $FS(\langle x_n\rangle_{n=1}^\infty)$ independent one must have $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A_1$ or $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A_2$.

In fact, Theorem 1.1 follows from an affirmative answer to the weaker Question 1.3.

1.3 Question. *Let G be a graph with vertices in $\mathbb{N}$ which contains no triangles. Must there exist a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\} \notin E(G)$ whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ with $F \cap H = \emptyset$?*

To see that an affirmative answer to Question 1.3 implies Theorem 1.1, let the graph G be defined exactly as above and let $\langle x_n\rangle_{n=1}^\infty$ be as guaranteed by an affirmative answer to the question. Suppose one has $F, H \in \mathcal{P}_f(\mathbb{N})$ with $\Sigma_{n\in F}\, x_n \in A_1$ and $\Sigma_{n\in H}\, x_n \in A_2$. Pick $k > \max(F \cup H)$. Then either $\{\Sigma_{n\in F}\, x_n, x_k\} \in E(G)$ or $\{\Sigma_{n\in H}\, x_n, x_k\} \in E(G)$.

On hearing Question 1.2, Erdős "retaliated" (his word) by asking the following much weaker question.

1.4 Question. *Let G be a triangle-free graph with vertices in $\mathbb{N}$. Must there exist $x \neq y$ such that $\{x, y, x+y\}$ is an independent set?*

In [7], this question was answered in the affirmative in the following strong fashion. Here K_m is the complete graph on m vertices.

1.5 Theorem. *Let G be a graph with vertex set $\mathbb{N}$ and assume there is some $m \in \mathbb{N}$ such that G contains no K_m. Then for each $\ell \in \mathbb{N}$, there is a finite sequence $\langle x_n\rangle_{n=1}^\ell$ such that $FS(\langle x_n\rangle_{n=1}^\ell)$ is an independent set.*

In [7] it was also shown that one cannot weaken the hypothesis of Theorem 1.5 to the assertion that G contains no K_ω (where K_ω is the complete graph on countably many vertices). For if $E(G) = \big\{\{x,y\} : x < y < 2x\big\}$ then G contains no K_ω, but given any $x < y$ in $\mathbb{N}$, one has $\{y, x+y\} \in E(G)$.

In Section 2 of this paper we answer Question 1.2 in the negative by exhibiting a triangle-free graph on $\mathbb{N}$ so that every $FS(\langle x_n\rangle_{n=1}^\infty)$ induces at least one edge in the graph.

In Section 3 we provide a strong affirmative answer to Question 1.3. That is, we show that if G is a graph with vertices in $\mathbb{N}$ and there exists some $m \in \mathbb{N}\backslash\{1,2\}$ such that G contains no K_m, then there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ and $F \cap H = \emptyset$, one has that $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\} \notin E(G)$.

In fact we show that the answer to Question 1.3 remains affirmative when the semigroup $(\mathbb{N}, +)$ is replaced by an arbitrary semigroup $(S, +)$. We write the semigroup