Minimal resolutions of algebras

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Abstract. A method is described for constructing the minimal projective resolution of an algebra considered as a bimodule over itself. The method applies to an algebra presented as the quotient of a tensor algebra over a separable algebra by an ideal of relations which is either homogeneous or admissable (with some additional finiteness restrictions in the latter case). In particular, it applies to any finite dimensional algebra over an algebraically closed field. The method is illustrated by a number of examples, viz. truncated algebras, monomial algebras and Koszul algebras, with the aim of unifying existing treatments of these in the literature.
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1. Introduction

A projective resolution of an algebra Λ, considered as a bimodule over itself, is fundamental in governing the homological properties of the algebra. Such a resolution may be used to compute Hochschild homology and cohomology, to provide functorial projective resolutions of one-sided modules and to compute the derived functors of $\mathrm{Hom}_\Lambda(-,-)$ and $- \otimes_\Lambda -$. Any two such resolutions are homotopic, but, when Λ admits a minimal resolution, then this resolution is unique up to isomorphism and should give the most natural and efficient method for making the computations already mentioned. Of course, the minimal resolution should also be an inherently interesting invariant of Λ. It is easy to see, for example, that the length of the minimal resolution is precisely the global dimension of Λ.

The primary aim of this paper is to construct the minimal projective bimodule resolution of an algebra Λ which is presented as a quotient of a tensor algebra over a separable algebra. The principal motivating example of such a tensor algebra is the path algebra of a quiver. However, the general formalism of tensor algebras provides a broader and more natural context for our results. Hence, throughout the paper, we fix a base field k, a k-separable algebra S and an S,S-bimodule X. We let Γ be the tensor algebra of X over S and J be the augmentation ideal in Γ generated by X. Without loss of generality, all quotient algebras $\Lambda = \Gamma/I$ will be taken over ideals $I \subseteq J^2$. Almost all spaces we encounter will be at least S,S-bimodules, which form a semi-simple category, precisely because S is separable. This category also has tensor products and the symbol $\otimes$ without a subscript will always denote tensor product over S.

Eilenberg's general theory of perfect categories [**10**] provides us with two classes of quotient algebras $\Lambda = \Gamma/I$ for which minimal bimodule resolutions exist: firstly, semi-primary algebras for which the relation ideal I is 'admissable', that is, contains a power of J; and, secondly, graded algebras for which I is homogeneous, with respect to the obvious grading of Γ by the tensor powers of X. We will restrict our attention to algebras in one or other of these classes.

The individual terms in the minimal resolution are known (see for example [**12**] §1.5 when Λ is finite dimensional). Indeed, a slight refinement of Eilenberg's original theory identifies the m-th term as

$$\Lambda \otimes \mathrm{Tor}_m^{\Lambda}(S,S) \otimes \Lambda$$

where S is regarded as a Λ-module on either side via the augmentation map $\Lambda \to S$ with kernel J/I. Furthermore, we may obtain a description in terms of the ideals I and J in Γ using the formulae (cf. [**7**])

$$\mathrm{Tor}_{2n}^{\Lambda}(S,S) \cong \frac{I^n \cap JI^{n-1}J}{JI^n + I^nJ} \qquad\qquad \mathrm{Tor}_{2n+1}^{\Lambda}(S,S) \cong \frac{JI^n \cap I^nJ}{I^{n+1} + JI^nJ} \tag{1.1}$$

Our main objective is to describe the maps in the resolution in terms of data associated to Γ and in a manner compatible with (1.1).

This objective can readily be achieved at the beginning of the resolution, which is known to have the form

$$0 \to I/I^2 \xrightarrow{c} \Lambda \otimes X \otimes \Lambda \xrightarrow{d} \Lambda \otimes \Lambda \xrightarrow{e} \Lambda \to 0 \tag{1.2}$$

The maps d and e are the reductions modulo I (on both sides) of the corresponding maps d_Γ and e_Γ in the basic structure sequence for any tensor algebra (see [**9**] Prop. 2.2.6)

$$0 \to \Gamma \otimes X \otimes \Gamma \xrightarrow{d_\Gamma} \Gamma \otimes \Gamma \xrightarrow{e_\Gamma} \Gamma \to 0 \tag{1.3}$$

Explicitly, e_Γ is the multiplication map and $d_\Gamma : 1 \otimes x \otimes 1 \mapsto x \otimes 1 - 1 \otimes x$. The connecting map c in (1.2) is induced by the restriction to I of the universal derivation

$$\Delta : \Gamma \to \Gamma \otimes X \otimes \Gamma : x_1 \cdots x_r \mapsto \sum_{i=1}^{r} x_1 \cdots x_{i-1} \otimes x_i \otimes x_{i+1} \cdots x_r$$

The basic strategy of the paper involves the construction of a series of four term exact sequences

$$0 \to \frac{I^{n+1}}{I^{n+2}} \xrightarrow{c_n} \Lambda \otimes \frac{JI^n \cap I^nJ}{JI^nJ} \otimes \Lambda \xrightarrow{d_n} \Lambda \otimes \frac{I^n}{JI^n + I^nJ} \otimes \Lambda \xrightarrow{e_n} \frac{I^n}{I^{n+1}} \to 0 \tag{1.4}$$

of which (1.2) is the case $n = 0$. The sequences (1.4) can then be spliced together to form a projective resolution of Λ, which we shall refer to as the 'spliced resolution'. Comparing (1.4) to (1.1), it is clear that this resolution is not necessarily minimal, but that the terms are closely related to the required minimal terms. The excess summands occur in naturally isomorphic pairs across the splicing

$$\Lambda \otimes \frac{I^n}{I^n \cap JI^{n-1}J} \otimes \Lambda \;\cong\; \Lambda \otimes \frac{I^n + JI^{n-1}J}{JI^{n-1}J} \otimes \Lambda$$