Stable module categories and their representation type

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Given a finite dimensional algebra Λ over an algebraically closed field one frequently disregards the projective objects in the category $\operatorname{mod}\Lambda$ of finite dimensional Λ-modules and focuses on the stable category $\underline{\operatorname{mod}}\,\Lambda$. The objects of $\underline{\operatorname{mod}}\,\Lambda$ are those of $\operatorname{mod}\Lambda$ but the Hom-groups are the ordinary Hom-groups modulo the subgroup of those morphisms which factor through a projective Λ-module.

In this paper we show that $\underline{\operatorname{mod}}\,\Lambda$ determines the representation type of Λ. Recall that the algebra Λ is either tame, i.e. all finite dimensional indecomposable Λ-modules belong to one-parameter families, or Λ is wild, i.e. there are two-parameter families of finite dimensional indecomposable Λ-modules [8]. Of course, one feels that this dichotomy should not depend on the deletion of finitely many objects in the category $\operatorname{mod}\Lambda$, and this is precisely one of the main results of this paper. More precisely, given another algebra Γ and an equivalence $\underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Gamma$, then Γ is tame if Λ is tame. Moreover, we show that the equivalence sends the one-parameter families in $\operatorname{mod}\Lambda$ to one-parameter families in $\operatorname{mod}\Gamma$. The fact that $\underline{\operatorname{mod}}\,\Lambda$ determines the representation type of Λ also follows, for some classes of symmetric algebras, from recent work of Assem, de la Peña and Erdmann [2, 9]; however their methods are completely different.

Equivalences between stable module categories have been studied by many authors. They naturally occur for instance in representation theory of finite groups. Another source of examples, which includes every algebra of Loewy length 2, is the class of algebras stably equivalent to a hereditary algebra. Usually the analysis concentrates on homological properties of the category $\operatorname{mod}\Lambda$ which are preserved by an equivalence $\underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Gamma$. In this paper we follow a different approach. We investigate pure-injective modules which are not necessarily finitely presented. Among them the endofinite modules are of particular interest. Recall that a module is endofinite if it is of finite length when regarded in the natural way as a module over its endomorphism ring. In order to study the non-finitely presented Λ-modules we introduce a new category $\varinjlim \underline{\operatorname{mod}}\,\Lambda$ which is essentially determined by the following three properties:

(i) $\varinjlim \underline{\operatorname{mod}}\,\Lambda$ contains, up to equivalence, $\underline{\operatorname{mod}}\,\Lambda$ as a full subcategory.

(ii) $\varinjlim \underline{\operatorname{mod}}\,\Lambda$ is an additive category with direct limits.

(iii) Every object in $\varinjlim \underline{\operatorname{mod}}\,\Lambda$ is a direct limit of objects in $\underline{\operatorname{mod}}\,\Lambda$.

The canonical functor $\operatorname{mod}\Lambda \to \underline{\operatorname{mod}}\,\Lambda$ induces a functor $\operatorname{Mod}\Lambda \to \varinjlim \underline{\operatorname{mod}}\,\Lambda$, and we use it to establish a close relationship between the endofinite objects in $\operatorname{Mod}\Lambda$ and $\varinjlim \underline{\operatorname{mod}}\,\Lambda$.

The representation type of $\underline{\mathrm{mod}}\,\Lambda$ then can be derived from a theorem of Crawley-Boevey which states that Λ is tame if and only if the endomorphism ring of every indecomposable endofinite Λ-module is a PI-ring [5].

The second part of this paper is devoted to the Auslander-Reiten translation $M \mapsto D\,\mathrm{Tr}\,M$ which Auslander and Reiten introduced in [3] for finitely presented modules over an artin algebra Λ, and which plays a prominent role in modern represenation theory. We extend this to an operation on the category of all Λ-modules and show that most of the basic properties for finitely presented modules carry over to arbitrary modules. Moreover, finiteness conditions like pure-injectivity and endofiniteness are preserved. Finally we show that the Auslander-Reiten translation induces a homeomorphism between the set of non-projective points and the set of non-injective points in the Ziegler spectrum of Λ. As a consequence we obtain that for a tame algebra every generic module remains fixed under the Auslander-Reiten translation. Recall that a module is generic if it is indecomposable endofinite but not finitely presented. Therefore our result complements Crawley-Boevey's theorem which states for a tame algebra that for every natural n almost all finitely presented indecomposable modules of endolength n are fixed under the Auslander-Reiten translation [4].

The final section of this paper is devoted to a remarkable property of certain triangulated categories which follows from our discussion of stable categories. We show that for every quasi-Frobenius ring Λ of finite representation type the triangulation on $\underline{\mathrm{mod}}\,\Lambda$ induces a triangulation on $\varinjlim \underline{\mathrm{mod}}\,\Lambda$. We obtain therefore non-trivial examples of triangulated categories having the property that the triangulation induces a triangulation on the category of ind-objects in the sense of Grothendieck and Verdier [11]

1. Locally finitely presented categories

We recall some terminology and some well-known facts about locally finitely presented categories. Most of this material can be found in Crawley-Boevey's exposition [7]. Let $\mathcal{A}$ be an additive category with direct limits. An object X in $\mathcal{A}$ is *finitely presented* provided that the representable functor $\mathrm{Hom}(X,\ )$ commutes with direct limits and we denote by $\mathrm{fp}(\mathcal{A})$ the full subcategory of finitely presented objects in $\mathcal{A}$. The category $\mathcal{A}$ is said to be *locally finitely presented* if the isomorphism classes of $\mathrm{fp}(\mathcal{A})$ form a set and every object in $\mathcal{A}$ is a direct limit of objects in $\mathrm{fp}(\mathcal{A})$.

Suppose now that $\mathcal{C}$ is a skeletally small additive category. We shall construct a locally finitely presented category $\varinjlim \mathcal{C}$ which is essentially determined by the following three