Finitistic dimension and Ziegler spectrum [1]

Henning Krause

Abstract: Given a two-sided artinian ring Λ, it is shown that the Ziegler spectrum of Λ forms a test class for certain homological properties of Λ. We discuss the finitistic dimension of Λ, Nunke's condition, and also the relation between the big and the little finitistic dimension.

Let Λ be a two-sided artinian ring. Denote by $\operatorname{Mod}\Lambda$ the category of (right) Λ-modules and by $\operatorname{mod}\Lambda$ the full subcategory of all finitely presented Λ-modules. Given a Λ-module M, we denote by $\operatorname{pd} M$ its projective dimension, and the *finitistic dimension* $\operatorname{Fd}\Lambda$ of Λ is the supremum of the projective dimensions of the Λ-modules with finite projective dimension. The *Ziegler spectrum* $\operatorname{Zsp}\Lambda$ of Λ is by definition the set of isomorphism classes of indecomposable pure-injective Λ-modules.

The aim of this note is to show that the Ziegler spectrum forms a test class for certain homological properties of Λ. Furthermore, the Ziegler spectrum carries a topology and we shall use its compactness to obtain an equivalent formulation of Nunke's condition. In the final part of this note we study the relation between $\operatorname{Fd}\Lambda$ and the little finitistic dimension $\operatorname{fd}\Lambda$ of Λ. Our first result is easily stated as follows.

Theorem 1 $\operatorname{Fd}\Lambda = \sup\{\operatorname{pd} M \mid M \in \operatorname{Zsp}\Lambda \text{ and } \operatorname{pd} M < \infty\}$.

We postpone the proof and first discuss some related homological properties of Λ. Suppose that Λ has a self-duality D between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. We are interested in *Nunke's condition* [8]:

$$\textstyle\coprod_{i\geq 0} \operatorname{Ext}^i_\Lambda(D\Lambda, M) \neq 0 \text{ for every } \Lambda\text{-module } M \neq 0.$$

To this end let us define the *Nunke dimension* of Λ as follows:

$$\operatorname{Nd}\Lambda = \inf\{d \in \mathbb{N}_0 \mid \textstyle\coprod_{i=0}^{d} \operatorname{Ext}^i_\Lambda(D\Lambda, M) \neq 0 \text{ for all } M \neq 0\}.$$

The following relation between Nunke dimension and finitistic dimension is essentially due to Jans [9], see also [4].

Proposition 2 $\operatorname{Nd}\Lambda \leq \operatorname{Fd}\Lambda$.

Proof: We use the fact that the functor

$$\operatorname{Inj}\Lambda \longrightarrow \operatorname{Proj}\Lambda, \qquad X \mapsto \operatorname{Hom}_\Lambda(D\Lambda, X)$$

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is an equivalence between the full subcategories of injective and projective Λ-modules. Suppose now that $\operatorname{Nd}\Lambda \geq n$ and choose a non-zero Λ-module M with $\coprod_{i=0}^{n-1} \operatorname{Ext}^i_\Lambda(D\Lambda, M) = 0$. Take an injective coresolution

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{\psi_0} I_1 \xrightarrow{\psi_1} I_2 \xrightarrow{\psi_2} \ldots$$

and put $N = \operatorname{Coker}\operatorname{Hom}_\Lambda(D\Lambda, \psi_{n-1})$. It is easily checked that $\operatorname{pd} N = n$. Thus $\operatorname{Fd}\Lambda \geq n$.

Remark Given any $n \in \mathbb{N}_0$, there is a finite dimensional algebra Λ over a field (with $\operatorname{rad}^2\Lambda = 0$ and $\operatorname{Nd}\Lambda = 2$) such that $\operatorname{Fd}\Lambda - \operatorname{Nd}\Lambda = n$ [6].

Theorem 3 $\operatorname{Nd}\Lambda = \inf\{d \in \mathbb{N}_0 \mid \coprod_{i=0}^{d} \operatorname{Ext}^i_\Lambda(D\Lambda, M) \neq 0$ *for all* $M \in \operatorname{Zsp}\Lambda\}$ *and Nunke's condition holds if and only if* $\operatorname{Nd}\Lambda < \infty$.

Some further definitions are needed in order to give the proof of both theorems. An additive functor $F\colon \operatorname{Mod}\Lambda \to \operatorname{Ab}$ into the category of abelian groups is *finitely presented* if there is a presentation

$$\operatorname{Hom}_\Lambda(Y,\ ) \longrightarrow \operatorname{Hom}_\Lambda(X,\ ) \longrightarrow F \longrightarrow 0$$

with X and Y in $\operatorname{mod}\Lambda$. We denote by $\operatorname{Ker} F$ the full subcategory of Λ-modules M with $F(M) = 0$. A full subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ is *definable* if there is a family F_i, $i \in I$, of finitely presented functors such that $\mathcal{X} = \bigcap_{i \in I} \operatorname{Ker} F_i$. A definable subcategory is automatically closed under direct limits, products and pure submodules, and it can be shown that this property characterizes a definable subcategory [11]. The notion of a definable subcategory allows us to describe the topology on $\operatorname{Zsp}\Lambda$ which was first introduced by Ziegler in model theoretic terms [12].

Proposition 4 (1) *The assignment* $\mathcal{X} \mapsto \mathcal{X} \cap \operatorname{Zsp}\Lambda$ *defines a bijection between the definable subcategories of* $\operatorname{Mod}\Lambda$ *and the closed subsets of* $\operatorname{Zsp}\Lambda$.

(2) $\operatorname{Zsp}\Lambda$ *is a quasi-compact space.*

Proof: (1) This result is due to Crawley-Boevey [2] and he relies on work of Herzog [5] and Ziegler [12]. We refer to [11] for a proof which uses the localization theory for locally coherent categories developed in [10]. However, the inverse of the assignment $\mathcal{X} \mapsto \mathcal{X} \cap \operatorname{Zsp}\Lambda$ is easily constructed; it sends a closed subset $\mathcal{U}$ of $\operatorname{Zsp}\Lambda$ to $\bigcap_{F(\mathcal{U})=0} \operatorname{Ker} F$.

(2) See [12] or [10].

The next lemma shows that some relevant subcategories of $\operatorname{Mod}\Lambda$ are definable.