KOLYVAGIN'S METHOD FOR SHIMURA VARIETIES: NEW TYPE OF EULER SYSTEMS. I.

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1. Introduction.

1.1. The purpose of the present paper is to prove some important steps of a natural program of generalization of Kolyvagin's proof of finiteness of Tate-Shafarevich group of a Taniyama-Weil elliptic curve E of analytic rank 0 or 1 ([K1], [K2], referred below as "K-case") to the case of other Shimura varieties. Some intermediate results of the present paper are of its own importance. For example, analogs of Eichler - Shimura relations for Picard modular varieties X (4.4.1 and 4.4.3) apparently were never published. Euler systems constructed in (5.3) are of new type, because of (1.3.1).

This program is the following. Let X be a Shimura variety (a level is fixed) having a sufficiently large family of Shimura subvarieties V_i of codimension r defined over a number field K, $\bar{X}$ a smooth compact model of X and $\mathcal{M}$ an irreducible quotient motive of $H^{2r-1}(\bar{X})$. We denote by $CH^r(\bar{X}\otimes K)_0$ the group of cycles on $\bar{X}$ of codimension r defined over K, homologically equivalent to 0, modulo rational equivalence, and we denote by $E=H^{2r-1}(\mathcal{M}(r),\mathbb{Z}_l)$ the l-adic realization of $\mathcal{M}(r)$, so there exists the Abel-Jacobi map $\mathrm{cl}\,'\colon CH^r(\bar{X}\otimes K)_0\to H^1(K,E)$. We choose a cycle $\phi(V)\in CH^r(\bar{X}\otimes K)_0$ which is a formal linear combination of V_i. An element $\mathrm{cl}\,'(\phi(V))\in H^1(K,E)$ is an analog of a Heegner point $y\in E(K)$ of the K-case, we denote it by y as well. The final goal of the first part of this program is a proof of the following theorem (a rough statement; see (1.6.1) for the exact statement):

Theorem. If $\exists\ V$ described above such that $y\neq 0$ then $SH(K,E)$ is finite and the Selmer group $S_{l^\infty}(K,E)$ is generated by y.

This part was realized for quaternion Shimura curves in [KL].

To complete this program we have to prove an analog of Gross - Zagier theorem [GrZ] for the present case; this will give us a proof of an analog (according to [BK] and [B]) of the Birch - Swinnerton-Dyer conjecture for sub- (or quotient) motives of Shimura varieties.

The realization of this program in whole is a huge task, first of all, because the existence of the main objects, namely motives H^{2r-1} and $\mathcal{M}$, depends on the standard conjectures of Tate and Langlands. Further, the proof of an analog of Gross - Zagier theorem seems now to be out of human force; the only result that we can try to obtain is to prove that for some cases $y\neq 0$ (maybe modulo some

[1]Partially supported by G.Soros Foundation and by an AMS grant

reasonable assumptions), i.e. to give an evidence that the whole program is not vacuous. A method of [B], §3 can be used for this purpose; see also (1.8.3).

Most steps of the first part of this program are realized in the present paper for one type of Shimura varieties, namely Picard modular threefolds (see (1.5), (4.1) for the definition), denoted by X_P. Some of these steps are valid without any significant change for other types of Shimura varieties. Remaining steps of the proof are of purely technical nature and do not contain serious obstacles. We consider for simplicity only the case of rank 0 ([K1]), although a construction of Euler systems of order 2 is given in (5.5).

Section 1.2 below contains the plan of the present paper together with the plan of realization of the first part of the program. For each step of this plan it is indicated is the obtained proof complete or partial, and can it be applied to other types of Shimura varieties, or not.

In Section 1.3 we indicate some main obstacles that do not exist in K-case and which are got over in the present paper. Section 1.4 contains a list of parallel notations of the present paper and [Gr] which is apparently the best survey of K-case. In 1.5 we give more details on conjectures of Langlands and Serre which are used in the proof. 1.6 contains an exact statement of the theorem, in 1.7 it is explained why namely X_P is chosen for the realization of this program, in 1.8 we describe some possibilities of further research and interesting unsolved questions.

1.2. The whole method of the present paper can be apparently applied to all Shimura varieties satisfying the following condition:

(1.2.1) X contains a family of subvarieties V_i of codimension r such that the reflex field of V_i is not contained or equal to the one of X.

Although throughout the whole paper we work mainly with X_P, proofs for some steps are valid for some others types of Shimura varieties practically without any change. There are 3 cases:

(A) A proof can be obtained for all Shimura varieties satisfying (1.2.1) but each type of these varieties claims its own calculations of growing volume;

(B) The proof is valid for all odd-dimensional Picard modular varieties; for other types of Shimura varieties we have case (A);

(C) The proof is valid for all types of Shimura varieties satisfying (1.2.1).

In Section 2 we remember the definition of Selmer and Tate - Shafarevich group of Galois modules and the definition of Abel-Jacobi map ([BK], [Fl], [J]).

Section 3 contains some information about Shimura varieties of PEL-type ([Sh1], [D]) that is necessary for us.

Step 1. It is shown by explicit construction of subvarieties that Picard modular varieties satisfy (1.2.1). Case B. Namely, in Section 4 we consider properties of a threefold $X = X_P$ and curves $V \subset X$. In (4.1) we define a curve $V = V_d \subset X$ which is an analog of a Heegner point in K-case ($d \in \mathbb{Z}$ is a non-essential parameter).

Step 2. Let $\mathbb{T}_p$ be the p-Hecke algebra of X. We choose an element $T_p \in \mathbb{T}_p$ and we consider it as a correspondence on X. The problem is to study the Galois action on irreducible components of $T_p(V)$, these are analogs of Kolyvagin's "trace relations"((4.3.4), (4.3.6), (5.3.8) (1), (2)). Case A.

The proof of this result (proposition 4.3.2) is highly calculational and claims consideration of many cases. Not all cases are considered in the present paper, by only reason of voluminous calculation (there is no obstacles of principle). By