A NOTE ON THE MODALITY OF PARABOLIC SUBGROUPS

Gerhard Röhrle

Abstract. For an algebraic group R acting morphically on an algebraic variety X the modality of the action, $\operatorname{mod}(R : X)$, is the maximal number of parameters on which a family of R-orbits on X depends upon.

Let G be a simple algebraic group defined over an algebraically closed field K of characteristic 0. Let P be a parabolic subgroup of G. Then P acts on its unipotent radical P_u via conjugation. The modality of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : P_u)$.

Let r and s be the semisimple rank of G and P respectively. We show that there is a quadratic polynomial f with rational coefficients such that the modality of P is at least $f(r - s)$. In particular, the modality of a Borel subgroup B of G grows at least quadratically with r. As a consequence, we obtain a finiteness result for algebraic groups from [8]: there is only a finite number of simple algebraic groups admitting parabolic subgroups of prescribed semisimple rank and prescribed modality. Combining our lower bounds with upper bounds from [6], we can compute the modality of Borel subgroups in some small rank cases. Here $\operatorname{mod} B$ agrees with our lower bounds.

1. Introduction

Let G be a reductive algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Let P be a parabolic subgroup of G. By P_u we denote the unipotent radical of P. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on P_u, $\operatorname{mod}(P : P_u)$, is the maximal number of parameters a family of P-orbits on P_u depends upon. Likewise for $\operatorname{mod}(P : \mathfrak{p}_u)$. Since the characteristic is zero, the exponential mapping is a P-equivariant morphism between $\mathfrak{p}_u$ and P_u. Thus we have $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$. The modality of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : P_u)$.

In [11, Theorem 3.1] it is shown that $\operatorname{mod} P$ is positive provided the nilpotent class of P_u is sufficiently large - at least 5 or 6, depending on the type of P. In the present paper we provide some explicit lower bounds for $\operatorname{mod} P$. These turn out to be quadratic polynomials in the difference of the semisimple ranks of G and P. We observe that the degree of these polynomials is optimal. In particular, the modality of a Borel subgroup of G grows quadratically with the rank of G. As a consequence, we obtain a finiteness result for simple algebraic groups admitting parabolic subgroups of a fixed modality and fixed semisimple rank from [8]. With the aid of these lower bounds for $\operatorname{mod} P$ we also obtain some refinements of the results from [11]. Combining our lower bounds with some upper bounds from [6], we can determine the modality of Borel subgroups in some small rank cases. It turnes out that in these instances $\operatorname{mod} B$ agrees with our lower bounds.

2. Notation and Recollection

For an algebraic group R we denote its Lie algebra by $\operatorname{Lie} R$ or simply by the corresponding lower case Gothic letter $\mathfrak{r}$. The identity component of R is denoted by R^0 and its unipotent radical by R_u. The Lie algebra of R_u is denoted by $\mathfrak{r}_u$. Throughout the identity element of an algebraic group is denoted by e. For the rank and semisimple rank of R we write $\operatorname{rank} R$ and $\operatorname{rank}_{ss} R$ respectively.

The descending central series $\{C^iR \mid i \geq 0\}$ of R is defined as usual: $C^0R := R$ and $C^{i+1}R := [C^iR, R]$ for $i \geq 0$. If $C^iR = \{e\}$ for some $i \geq 0$, then R is nilpotent. In that case the *length* of the descending central series of R, denoted by $\ell(R)$, is the smallest integer m such that $C^mR = \{e\}$, i.e., $\ell(R) = m$; equivalently, R is nilpotent of class m.

Recall from [9] that for an algebraic group R acting morphically on an algebraic variety X the *modality* of the action, $\operatorname{mod}(R : X)$, is defined as

$$\operatorname{mod}(R : X) = \max_Z \min_{z \in Z} \operatorname{codim}_Z R^0 \cdot z,$$

where Z runs through all irreducible R^0-invariant subvarieties of X. Clearly, $\operatorname{mod}(R : X) \geq \dim X - \dim R$. In case X is an irreducible variety let $K(X)^R$ denote the field of R-invariant rational functions on X. Since $\min_{x \in X} \operatorname{codim}_X R \cdot z = \operatorname{trdeg} K(X)^R$ (e.g. see [10, 2.3]), $\operatorname{mod}(R : X)$ measures the maximal number of parameters on which a family of R-orbits on X depends upon. The modality of the action of R on X is 0 precisely when R admits only a finite number of orbits on X. The notion of modality originates in the work of Arnold [1]; see also Vinberg [13] and [10, 5.2]. We call $\operatorname{mod}(P : P_u)$ the *modality* of P.

Let G be a connected reductive algebraic group, T a maximal torus in G and Ψ the set of roots of G with respect to T. Fix a set of simple roots Π of Ψ and let B be the Borel subgroup of G corresponding to Π. A parabolic subgroup of G containing B is called a standard parabolic subgroup. For a subset J of Π we denote by P_J the standard parabolic subgroup corresponding to J such that $P_\varnothing = B$.

As every parabolic subgroup P of G is conjugate to a standard one, and since $\ell(P_u)$ and $\operatorname{mod} P$ are invariant under conjugation by elements from G, we may if we wish assume that the parabolic subgroups under consideration are standard.

As a general reference for algebraic groups we cite Borel's book [2] and for information on root systems we refer the reader to Bourbaki [3]. The simple roots in a base of a root system of a simple algebraic group are indexed in accordance with *loc. cit.* (Planches I - IX).

We follow [5] by calling a closed connected subgroup H of G *regular* if the normalizer of H in G contains a maximal torus T of G. In that case the root spaces of $\mathfrak{h}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of H with respect to T, $\Psi(H)$, is a subset of Ψ, the root system of G relative to T, and is closed under addition in Ψ. If H is a reductive regular subgroup of G, then $\Psi(H)$ is a subsystem of Ψ. Conversely, by choosing a subsystem $\Psi(H)$ of Ψ, or equivalently, a set of simple roots $\Pi(H)$ for $\Psi(H)$, we can define a connected semisimple regular subgroup H of G whose root system is precisely $\Psi(H)$. Moreover, a set of simple roots $\Pi(H)$ for $\Psi(H)$ can always be chosen such that the Borel subgroup of H relative to $\Pi(H)$ is precisely $H \cap B$, where B is the Borel subgroup of G relative to Π. In particular, $\Psi^+(H) \subseteq \Psi^+$. Likewise, we speak of regular subalgebras.