A Topological Study of the Fractional Fibonacci Groups

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Abstract

We consider the fractional generalization of the Fibonacci groups $F(2, 2n)$, which is naturally from a topological point of views. The topological properties of the arising three-dimensional manifolds are studied. In particular, these manifolds are described in terms of the Dehn surgery, as the two-fold branched coverings of the three-dimensional sphere and as the cyclic coverings of the three-dimensional sphere, branched over the two-bridge knots.

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1 The Fractional Fibonacci Groups

The present article is devoted to some generalization of the Fibonacci groups and studying of geometric and topological properties of the corresponding three-dimensional closed orientable manifolds.

The Fibonacci groups $F(2, m)$ were introduced by J. Conway [5] and have the following presentations:

$$F(2,m) = \langle x_1, x_2, \ldots, x_m \mid x_i \, x_{i+1} = x_{i+2}, \quad i = 1, \ldots, m \rangle, \tag{1}$$

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where the indices are taken $mod\, m$. The nomenclature follows from the inductive definition of the Fibonacci numbers with initials $a_1 = a_2 = 1$:

$$a_i + a_{i+1} = a_{i+2}. \tag{2}$$

The studying of the Fibonacci groups was inspired by the Conway's question: whether or not the groups $F(2,m)$ are finite? The answer was given due to [6], [9], [18], and [21] (a nice survey can be found in [22]) : the group $F(2,m)$ is finite precisely when $m = 1,2,3,4,5,7$.

In [12] D. L. Johnson introduced the following generalization of the Fibonacci groups:

$$F(r,m) = \langle x_1, x_2, \ldots, x_m \mid x_i \cdots x_{i+r-1} = x_{i+r}, \quad i = 1, \ldots, m \rangle, \tag{3}$$

for a given integer $r \geq 2$, where the indices are taken $mod\, m$. The properties of $F(r,m)$ were studied in [13] (see [22] for a survey, where the knowledge about the groups $F(r,m)$, $2 \leq r \leq 10$ and $2 \leq n \leq 10$, is summarized).

Another generalization of the Fibonacci groups were introduced by C. Maclachlan [14]. For an integer $k \geq 1$ he considered a group

$$F^k(2,m) = \langle x_1, x_2, \ldots, x_m \mid x_i\, x_{i+1}^k = x_{i+2}, \quad i = 1, \ldots, m \rangle, \tag{4}$$

where the indices are taken $mod\, m$. One can see the correspondence between this generalization of the Fibonacci group and the following generalization $\{a_i^k\}$ of the Fibonacci numbers :

$$a_i^k + k\, a_{i+1}^k = a_{i+2}^k, \tag{5}$$

where the inductive law depends on a given integer parameter k.

It is natural to generalize the above inductive law to the fractional parameter case. More exactly, let us consider a "fractional" analogue $\{a_i^{k/l}\}$ of the Fibonacci numbers with the following inductive law:

$$a_i^{k/l} + \frac{k}{l}\, a_{i+1}^{k/l} = a_{i+2}^{k/l} \tag{6}$$

for a fractional number $\frac{k}{l}$. By the same arguments, it is natural to consider groups with the following presentations:

$$F^{k/l}(2,m) = \langle x_1, x_2, \ldots, x_m \mid x_i^l\, x_{i+1}^k = x_{i+2}^l, \quad i = 1, \ldots, m \rangle \tag{7}$$

for coprime integers k and l, and an integer $m \geq 3$, where the indices are taken $mod\, m$. A group $F^{k/l}(2,m)$ is said to be *a Fractional Fibonacci group*.

One of the aims of the present article is to demonstrate that the extension of the Fibonacci groups $F(2,m)$ to the Fractional Fibonacci groups $F^{k/l}(2,m)$ is naturally from a topological point of views.