DIFFUSIONS ON PATH AND LOOP SPACES: EXISTENCE, FINITE DIMENSIONAL APPROXIMATION AND HÖLDER CONTINUITY

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1. Introduction and results

In his paper [K95], T. Kazumi gave a construction for the Ornstein–Uhlenbeck process on the path space over a compact Riemannian manifold by finite dimensional approximation. He also derived regularity properties of the sample paths. A major ingredient in his proof is an estimate for the Ornstein–Uhlenbeck operator (resp. its finite–dimensional projections) applied to certain distance functions. This estimate, which is used to prove tightness of the laws of the approximating finite–dimensional diffusions, seems to rely heavily on the special situation considered. In particular, it is not obvious how to carry over the result to Ornstein–Uhlenbeck operators on *loop* spaces. In fact, in contrast to first order objects, second order objects on path spaces over manifolds are rather complicated.

On the other hand, there is a well–known method for proving tightness and Hölder–continuity based on the Lyons–Zheng decomposition (cf. [LZhe88], [LZ94]), which only involves estimations of first order derivatives. This method has been successfully applied to approximate Hilbert–space– and $C([0,1],\mathbb{R}^m)$–valued diffusions, cf. [RZ95], [LRZ95].

In this paper, we describe two approaches to existence, finite-dimensional approximation, and Hölder continuity of symmetric diffusions on path and loop spaces over manifolds, cf. Theorems 1 and 2 below. Both apply not only to Ornstein–Uhlenbeck processes, but also to more general diffusions, provided an easy to check condition holds (cf. (A 1), (A 2) resp. (A 3) below, see also the examples in Section 5). Our first approach (Theorem 1) is somehow similar to Kazumi's proof for the Ornstein–Uhlenbeck operator on path space, but it is based on estimates following from the Lyons–Zheng decomposition. The point of view taken here is to look at path–space–valued processes as manifold–valued two–parameter processes. We obtain both existence/approximation and Hölder continuity results. If one is not interested in Hölder continuity but only existence and finite dimensional approximation, it can be more convenient to view path–space–valued processes as infinite dimensional rather than two–parameter processes. This is carried out in our second approach, leading to a stronger approximation result (Theorem 2), which contains one of the main results from [LRZ95] as a special case. Our method of proof here is partially motivated by an idea of K. Kuwae and T. Uemura [KuwU95].

To state our results in detail, let M be a connected submanifold of $\mathbb{R}^m$, $m \in \mathbb{N}$. We assume that M carries a Riemannian metric g such that

$$K^{-1}|v| \le g(v,v)^{1/2} \le K|v| \quad \text{for all } v \in TM \tag{1.1}$$

for some constant $K \in [1,\infty)$. This is always the case if (M,g) is a compact Riemannian manifold embedded into $\mathbb{R}^m$ by Whitney's theorem. Let d be the corresponding distance function. We assume that M is *complete* w.r.t. d.
The space $C([0,1],M)$ of continuous paths on M with supremum distance

$$d_\infty(\omega,\omega') = \sup_{s\in[0,1]} d(\omega(s),\omega'(s))$$

may be viewed as an infinite–dimensional manifold. Suppose Ω is a subset of type

$$\Omega = \{\omega \in C([0,1],M);\ \varphi(\omega(0),\omega(1)) = 0\}$$

where $\varphi \in C^\infty(M\times M,\mathbb{R}^\ell), \ell \in \mathbb{N}$, is a fixed map.
Ω may be the space of paths resp. loops on M starting at a fixed point, or the space of all paths resp. loops. We fix a probability measure μ on Ω such that supp $\mu = \Omega$. Let

$$\begin{aligned}\mathcal{F}C_b^\infty(\Omega) := \{F:\Omega\to\mathbb{R};\ F(\omega) = f(\omega(s_1),\dots,\omega(s_n)) \quad \text{for some}\\ n\in\mathbb{N},\ s_1,\dots,s_n\in[0,1] \text{ and } f\in C_b^\infty(M^n)\}\end{aligned}$$

be the cylinder functions on Ω. Here $C_b^\infty(M^n)$ denotes the set of all smooth functions on M^n having bounded derivatives of any order.
Suppose we are given a quadratic form $\mathcal{E}$ on $\mathcal{F}C_b^\infty(\Omega)$ of type

$$\mathcal{E}(F,G) = \int \Gamma(F,G)\ d\mu\ ,$$

where $\Gamma : \mathcal{F}C_b^\infty(\Omega)\times\mathcal{F}C_b^\infty(\Omega)\to L^1(\Omega,\mu)$ is a symmetric bilinear operator satisfying $\Gamma(F,F)\ge 0$ μ–a.e., and the product rule

$$\Gamma(FG,H) = F\cdot\Gamma(G,H) + G\cdot\Gamma(F,H) \quad \text{for all } F,G,H\in\mathcal{F}C_b^\infty(\Omega)\ . \tag{1.2}$$

We also write $\mathcal{E}(F)$ and $\Gamma(F)$ instead of $\mathcal{E}(F,F)$ and $\Gamma(F,F)$. The main examples we have in mind are those where

$$\Gamma(F,G) = \langle DF, DG\rangle \tag{1.3}$$

for some kind of gradient D on Ω taking values in a suitable "generalized tangent bundle" to Ω, and some suitable "generalized metric" $\langle\cdot,\cdot\rangle$ on the tangent bundle. We refer to the examples in Section 5 below and to the extensive literature for the definitions of well–suited gradients for the various path and loop spaces, cf. e.g. [MalMal90], [DR92], [ALeR92], [FMal93]. Since the choice of the "metric" is left open,