# ON PARABOLIC SUBGROUPS OF CLASSICAL GROUPS WITH A FINITE NUMBER OF ORBITS ON THE UNIPOTENT RADICAL

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Abstract. Let $G$ be a classical algebraic group defined over an algebraically closed field of characteristic zero. In case the simple factors of $G$ are of type $A_r$, $B_r$ or $C_r$, all parabolic subgroups $P$ of $G$ with a finite number of orbits on the unipotent radical $P_u$ are determined: this is the case precisely when the class of nilpotency of $P_u$ is at most four. For groups of type $D_r$ we obtain partial results, depending on the structure of $P$.

## 1. Introduction

Let $G$ be a reductive algebraic group defined over an algebraically closed field $k$ of characteristic 0. Let $P$ be a parabolic subgroup of $G$ with unipotent radical $P_u$. The general problem of determining each parabolic subgroup $P$ of a reductive algebraic group $G$ with a finite number of orbits on $P_u$ was posed in [7]. The cases for Borel and semisimple rank one parabolic subgroups were classified in [6] and [7] respectively. In the case that the simple factors of $G$ are of type $A_r$, $B_r$, or $C_r$, it was shown in [10, Theorem 3.1] that the number of orbits of $P$ on $P_u$ is infinite provided the class of nilpotency of $P_u$ is at least 5. The converse of this statement which was conjectured in [10, 3.7] is proved in this paper.

Let $\ell(P_u)$ denote the nilpotency class of $P_u$. Our principal results are

**Theorem 1.1.** *Let $G$ be a simple classical algebraic group and $P$ a parabolic subgroup of $G$. The number of $P$-orbits on $P_u$ is finite provided*

(i) *$G$ is of type $A_r$, $B_r$ or $C_r$, and $\ell(P_u) \leq 4$;*
(ii) *$G$ is of type $D_r$ and $\ell(P_u) \leq 3$;*
(iii) *$G$ is of type $D_r$, $\ell(P_u) = 4$, and $P$ is stable under the automorphism of $G$ of order 2 stemming from the interchange of the simple roots $\alpha_{r-1}$ and $\alpha_r$.*

Combining Theorem 1.1 with Theorem 3.1 from [10] we immediately obtain

**Theorem 1.2.** *Let $G$ be a simple classical algebraic group and $P$ a parabolic subgroup of $G$. In the case that $G$ is of type $D_r$ assume that $P$ is invariant under the graph automorphism of $G$ of order two. Then the number of $P$-orbits on $P_u$ is finite if and only if $\ell(P_u) \leq 4$.*

*Remarks 1.3.* Theorems 1.1 and 1.2 naturally extend to reductive classical groups by means of their simple components. A detailed discussion of parabolic subgroups with an Abelian unipotent radical, including the structure of orbit closures, appears in [8]. Parabolic subgroups of classical groups with $\ell(P_u) = 2$ are treated in [11]. Results for exceptional groups can be found in [5].

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In Section 2 we first recall some results from [7] and [10] and apply them to our situation in Section 3 in showing that for a large class of parabolic subgroups $P$ of classical groups, the question of finiteness of the number of $P$-orbits on $P_u$ can be reduced to the case of groups of type $A_r$. In the final part we prove the statement of Theorem 1.2 for $GL(V)$, where $V$ is a finite-dimensional $k$-vector space, by translating the problem into a question of the representation-type of a particular category of $\Delta$-filtered modules over a quasi-hereditary algebra.

Let $G$ be a connected reductive algebraic group, $T$ a maximal torus in $G$ and $\Psi$ the set of roots of $G$ with respect to $T$. Fix a set of simple roots $\Pi$ of $\Psi$ and let $B$ be the Borel subgroup of $G$ corresponding to $\Pi$. A parabolic subgroup of $G$ containing $B$ is called standard. By $\Psi(P)$ we denote the set of roots of $P$. As every parabolic subgroup of $G$ is conjugate to a standard one, we may if we wish assume that the parabolic subgroups under consideration are standard.

Because a reductive algebraic group is the direct product of simple ones over its center, and similarly for parabolic subgroups, we can reduce to the case of a simple algebraic group in our study of parabolic subgroups with the desired finiteness property. Henceforth we assume that $G$ is a connected simple algebraic group.

Since $\operatorname{char} k = 0$, the exponential map is a $P$-equivariant isomorphism between the Lie algebra of $P_u$, denoted by $\mathfrak{p}_u$, and $P_u$. This allows us to work on $\mathfrak{p}_u$ instead of $P_u$, whenever convenient.

As a general reference for algebraic groups we cite Borel's book [1] and for information on root systems we refer the reader to Bourbaki [2]. Concerning the general theory of representations of finite-dimensional algebras we refer to the book by Gabriel and Roiter [4] and Ringel's monograph [9].

## 2. Some Recollections

We maintain the notation from the Introduction. The following result is part of Theorem 3.1 in [10].

**Theorem 2.1.** *Let $G$ be a simple classical algebraic group. Let $P$ be a parabolic subgroup of $G$. Then the number of $P$-orbits on $P_u$ is infinite provided*

(i) *$G$ is of type $A_r$, $B_r$ or $C_r$, and $\ell(P_u) \geq 5$;*
(ii) *$G$ is of type $D_r$ and $\ell(P_u) \geq 6$;*
(iii) *$G$ is of type $D_r$, $\ell(P_u) = 5$, and $P$ is stable under the automorphism of $G$ of order 2 stemming from the interchange of the simple roots $\alpha_{r-1}$ and $\alpha_r$.*

Let $\Theta$ be a semisimple automorphism of $G$ and let $P$ be a $\Theta$-stable parabolic subgroup of $G$. As $P_u$ is a characteristic subgroup of $P$, it is $\Theta$-stable as well. Let $G^\Theta$, $P^\Theta$, and $P_u^\Theta$ be the $\Theta$-fixed point subgroup of $G$, $P$, and $P_u$ respectively. Our next result is Corollary 2.8 from [7].

**Proposition 2.2.** *Let $\Theta$ be a semisimple automorphism of $G$. Assume that $P$ is a $\Theta$-stable parabolic subgroup of $G$. If $P$ has a finite number of orbits on $P_u$, then so does $P^\Theta$ on $P_u^\Theta$.*

*Remark 2.3.* For $P$ a standard parabolic subgroup of $G$, the class of nilpotency $\ell(P_u)$ is easily determined as follows, cf. [10, Lemma 2.5]. Let $\beta \in \Psi^+$. Write $\beta = \sum_{\alpha\in\Pi} c_\alpha(\beta)\alpha$, with $c_\alpha(\beta) \in \mathbb{Z}_0^+$ for each $\alpha \in \Pi$. Let $\rho$ be the highest root of $\Psi$. Then $\ell(P_u) = \sum_{\alpha\in\Pi\setminus\Psi(P)} c_\alpha(\rho)$.