Matrices whose inverses are generalized M–matrices

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Abstract

In 1972 T.L. Markham defined the so-called matrices of type D [M]. Markham proved that the inverse of a certain type D matrix is a tridiagonal M–matrix. Recently, several generalizations of this result were established, i.e. other classes of inverse M–matrices were introduced. Here we give another generalization. We consider block matrices whose blocks are hermitian positive semidefinite, and in some cases pairwise commute which leads to block type D matrices and block generalized ultrametric matrices. We will establish that the inverse of a block generalized ultrametric matrix is a generalized M–matrix. Moreover, we give an explicit formula for the inverse of a block type D matrix which shows that the inverse is a block tridiagonal generalized M–matrix.

1 Introduction and Preliminaries

Since Ostrowski introduced in 1937 the class of M–matrices, this class has been studied intensively and a lot of important properties of M–matrices have been established. Moreover, several generalizations of M–matrices have been introduced. Recently, Elsner and Mehrmann [EM] and Nabben [N1], [N2] defined the class of generalized M–matrices. These block matrices arise e.g. in the numerical solution of Euler- and Stokes equations (see [N2]).

Definition 1.1 *A block matrix $A = [A_{ij}] \in \mathbb{C}^{\mathrm{nm,nm}}$ with $A_{ij} = A_{ij}^H \in \mathbb{C}^{\mathrm{m,m}}$, for $1 \leq i, j \leq n$, and A_{ij} negative semidefinite for $i \neq j$, is called a generalized Z–matrix. If there exists a positive vector $d = [d_i] \in \mathbb{R}^{\mathrm{n}}$ such that*

$$\sum_{j=1}^{n} d_j A_{ij} \quad \textit{is positive definite for all } i$$

then A is called a generalized M–matrix.

If $m = 1$, then generalized Z–matrices are just Z–matrices and generalized M–matrices are M–matrices. Several equivalent definitions and properties of generalized M–matrices are given in [EM] and [N1] [N2].

An important problem related to M–matrices is the so called inverse M–matrix problem i.e. characterizing all nonnegative matrices whose inverses are M–matrices. In 1972 T.L. Markham introduced the first class of inverse M–matrices, the matrices of type D [M]. Recently this result is generalized in several directions. In [N3] matrices of type D with no restriction on the sign of the entries are discussed. It was shown that the inverses of such matrices are tridiagonal Z–matrices and one can deduce from the number of nonpositive diagonal entries the class of Z–matrix to which A^{-1} belongs. Moreover, McDonald, Neumann, Schneider, and Tsatsomeros [MNST] and Nabben and Varga [NV] introduced a new class of inverse M–matrices. This class of so called generalized ultrametric matrices contains the type D matrices and (strictly) ultrametric matrices.

Here we give another generalization of Markham's result [M] as well as generalizations of results of [MNST] and [NV]. We consider block matrices whose blocks are of the same size and hermitian positive semidefinite. If all blocks can be ordered by the positive definite ordering, we define block type D matrices and block generalized ultrametric matrices by replacing real nonnegative numbers with hermitian positive semidefinite matrices. We then show among other properties, that the inverses of these matrices are generalized M–matrices. For block type D matrices we will give an explicit formula for the inverse. Moreover we consider block matrices whose blocks commute. Matrices of this type arise e.g. in the numerical solution of the Laplace equation.

The paper is organized as follows. At the end of this section we mention some preliminaries and some notations. In Section 2 we consider first block matrices whose blocks commute. Then we introduce the class of block generalized ultrametric matrices. Section 3 deals with the class of block type D matrices.