ON TWO RESULTS ABOUT FIBRATIONS

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Abstract. Using equivariant methods, we provide straightforward proofs of a result of Chachólski and a result of Spivak about fibrations.

Introduction

Let $\tilde{h}_*$ denote a reduced homology theory on the category of based spaces,and let h_* denote the corresponding extension to an unreduced homology theory on the category of unbased spaces given by $h_*(X) = \tilde{h}_*(X_+)$, where X_+ denotes X with the addition of a disjoint basepoint.

The following was proven by W. Chachólski [2], using the elaborate machinery of 'cellular inequalities' developed by E. Dror Farjoun, et.al.:

Theorem A. *Suppose that*

$$F \to E \to B$$

is a fibration, with B a connected, based space. Assume that

$$h_*(F) \to h_*(E)$$

is an isomorphism for all degrees $*$. Then $\tilde{h}_*(B) = 0$.

Chacólski tells me that Dror Farjoun has asked if a 'classical proof' of Theorem A exists. The first part of this paper provides such a proof.

It is perhaps worth remarking here that if the fibration admits a section, the result is trivial, since the homology of B is a retract of the homology of the mapping cone of the inclusion $F \to E$ and the latter has trivial homology. This gives a clue as to how the result is to be proved in the general case: in effect, we will show that even when the fibration doesn't have a section a homological retraction still exists.

The second result which we give a new proof of was originally due to M. Spivak, reformulated by W. Browder ([3,4.3], [1,I.4.3]):

Theorem B. *Suppose that*

$$F \to E \xrightarrow{p} B$$

is a fibration, where B is 1-connected. Suppose also that the Thom isomorphism is satisfied, i.e., for some positive integer $n \geq 2$ there exists a class $u \in H^n(p)$ in singular homology for which

$$u\cap\colon H_*(p) \to H_{*-n}(B_+)$$

is an isomorphism for all $*$*. Then* F *is a homology* $(n-1)$*-sphere. In particular, if* F *is* 1*-connected, then* $F \simeq S^{n-1}$.

The Spivak-Browder proof of Theorem B was computational, making intricate use of the relative Serre spectral sequence.

In contrast, our methods of proving Theorems A and B will be to convert the statements into questions involving spectra equipped with the action of a topological group; this enables us to apply standard equivariant techniques.

Acknowledgment. I would Like to thank Wojciech Chachólski for explaining to me his proof of Theorem A.

Outline of paper. §1 is preliminary. The main result of that section is 1.5. §2 contains the proof of Theorem A. §3 contains the proof of Theorem B. §4 is an addendum which contains variants of Theorems A and B which we state without proof.

§1 Preliminaries

The spaces in this paper are to be given the compactly generated topology.

1.1. Spectra with G-action. Let $G.$ be a simplicial group. Then its realization $G := |G.|$ (of its underlying simplicial set) is a topological group object in the category of compactly generated spaces.

A *pre-spectrum with G-action* $\mathbf{W}$ is a collection of based (left) G-spaces $\mathbf{W}_n$, $n \in \{0, 1, ...\}$, and equivariant maps $\Sigma\mathbf{W}_n \to \mathbf{W}_{n+1}$ where G acts trivially in the suspension coordinate in the source. We say that $\mathbf{W}$ is a *spectrum with G-action* when the adjoint map $\mathbf{W}_n \to \Omega\mathbf{W}_{n+1}$ is a weak homotopy equivalence (of underlying topological spaces).

Given a pre-spectrum with G-action $\mathbf{W}$, we can form its *spectrification* $\bar{\mathbf{W}}$ by taking $\bar{\mathbf{W}}_n$ to be

$$\operatorname*{hocolim}_k \Omega^k \mathbf{W}_{n+k} \,.$$

This is a spectrum with G-action.

Given a based G-space Y, we can form the spectrum with G-action $\mathbf{W} \wedge Y$: it is defined to be the spectrification of the pre-spectrum with G-action whose n-th space is $\mathbf{W}_n \wedge Y$, with G-acting diagonally.

If $\mathbf{W}$ is a pre-spectrum with G-action, we can form its *homotopy orbit* spectrum $\mathbf{W}_{hG}$; it is given by spectrifying the pre-spectrum with G-action whose n-th space is the reduced Borel construction on $\mathbf{W}_n$, i.e., $EG_+ \wedge_G \mathbf{W}_n$.

By a map of (pre-)spectra $f\colon \mathbf{X} \to \mathbf{Y}$ with G-action, we mean a collection of G-maps $f_n\colon \mathbf{X}_n \to \mathbf{Y}_n$ such that f_{n+1} composed with the structure map $\Sigma\mathbf{X}_n \to \mathbf{X}_{n+1}$ coincides with the structure map $\Sigma\mathbf{Y}_n \to \mathbf{Y}_{n+1}$ composed with Σf_n.

Given a (pre)-spectrum $\mathbf{X}$, recall that its *homotopy groups* $\pi_j(\mathbf{X})$ for $j \in \mathbb{Z}$ are given by $\operatorname{colim}_{\,j+n}(\mathbf{X}_n)$. We say $\mathbf{X}$ that *r-connected* if its homotopy groups vanish below degree r. We say that $\mathbf{X}$ is *bounded below* if it is r-connected for some integer r.

If $\mathbf{X}$ is a spectrum with G-action, its homotopy groups are a left $\mathbb{Z}[\pi_0(G)]$-module. The action of $[g] \in \pi_0(G)$ on $\pi_*(\mathbf{X})$ is defined by applying homotopy groups to the multiplication by g map.