Simple connectedness of the 3-local geometry of the Monster

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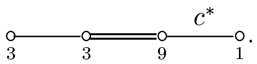
Abstract

We consider the 3-local geometry $\mathcal{M}$ of the Monster group M introduced in [BF] as a locally dual polar space of the group $\Omega_8^-(3)$ and independently in [RS] in the context of minimal p-local parabolic geometries for sporadic simple groups. More recently the geometry appeared implicitly in [DM] within the Z_3-orbifold construction of the Moonshine module $V^\natural$. In this paper we prove the simple connectedness of $\mathcal{M}$. This result makes unnecessary the refereeing to the classification of finite simple groups in the Z_3-orbifold construction of $V^\natural$ and realizes an important step in the classification of the flag-transitive c-extensions of the classical dual polar spaces (cf. [Yo]). We make use of the simple connectedness results for the 2-local geometry of M [Iv1] and for a subgeometry in $\mathcal{M}$ which is the 3-local geometry of the Fischer group $M(24)$ [IS].

1 Introduction

The Monster group M acts flag-transitively on a diagram geometry $\mathcal{M}$ which is described by the following diagram:



The elements of $\mathcal{M}$ corresponding to the nodes from the left to the right on the diagram are called *points*, *lines*, *planes* and *quadrics*, respectively. The residue of a quadric is the classical polar space associated with the group $\Omega_8^-(3)$. The quadrics and planes incident to a line form the geometry of vertices and edges of a complete graph on 11 vertices. The existence of $\mathcal{M}$ was independently established in [BF] and [RS]. We follow [BF] to review briefly the construction of $\mathcal{M}$ and to formulate its basic properties. The starting point is the description of conjugacy classes of the subgroups of order 3 in the Monster [At].

Lemma 1.1 *In the Monster group M every element of order 3 is conjugate to its inverse and there are exactly three conjugacy classes of subgroups of order 3 with representatives σ, μ and τ, so that*

(a) $N_M(\sigma) \sim 3^{1+12}.2{\cdot}Suz.2$, *where Suz is the Suzuki sporadic simple group;*

(b) $N_M(\mu) \sim 3{\cdot}M(24)$, *where $M(24)$ is the largest sporadic Fischer 3-transposition group;*

(c) $N_M(\tau) \sim Sym(3) \times F_3$, *where F_3 is the sporadic simple group discovered by Thompson.* □

We define a subgroup of order 3 in M to be of *Suzuki*, *Fischer* or *Thompson type* if it is conjugate to σ, μ or τ from 1.1, respectively.

A crucial role in the construction of $\mathcal{M}$ is played by a subgroup $M_8 \sim 3^8.\Omega_8^-(3).2$ in M. If $Q_8 = O_3(M_8)$ then M_8/Q_8 is extension of the simple orthogonal group $\Omega_8^-(3)$ by an automorphism of order 2, Q_8 is the natural orthogonal module for M_8/Q_8 and $N_M(Q_8) = M_8$.

Lemma 1.2 *Let φ be the orthogonal form of minus type on Q_8 preserved by M_8/Q_8. Then M_8/Q_8 acting on the subgroups of order 3 in Q_8 has two orbits I and N such that*

(a) $|I| = 1\,066$, *the subgroups in I are isotropic with respect to φ and of Suzuki type in M; for $\sigma \in I$ we have $N_{M_8/Q_8}(\sigma) \sim 3^6.2{\cdot}U_4(3).2^2$;*

(b) $|N| = 2\,214$, *the subgroups in N are non-isotropic with respect to φ and of Fischer type in M; for $\mu \in N$ we have $N_{M_8/Q_8}(\mu) \sim \Omega_7(3).2$.*

Proof: Under the action of $O^2(M_8/Q_8) \sim \Omega_8^-(3)$ the set of order 3 subgroups in Q_8 splits into three orbits I, N_1, N_2 with lengths 1 066, 1 107, 1 107 and stabilizers isomorphic to $3^6.2{\cdot}U_4(3).2$, $\Omega_7(3).2$, $\Omega_7(3).2$, respectively (cf. [At]). As 3^{17} divides the order of each of the stabilisers there are no Thompson type subgroups in Q_8 and as the elements of I are 3-central they are of Suzuki type. By Lagrange $3^7\Omega_7(3)$ is not involved in Suz and since 3^{1+12} has no elementary abelian subgroup or order 3^8, N_1 and N_2 consist of Fischer type subgroups. Finally, in $M(24)$ all subgroups of order 3^7 whose normalizer involve $\Omega_7(3)$ are conjugated. Hence N_1 and N_2 fuse into a single M_8/Q_8-orbit.□

By 1.2 the polar space acted on flag-transitively by M_8/Q_8 can be identified with the Suzuki-pure subgroups in Q_8 with two subgroups being incident if one of them contains the other one. Let Q_1, Q_2, Q_3 be Suzuki-pure subgroups in Q_8 with $Q_1 < Q_2 < Q_3$, so that $|Q_i| = 3^i$ for $1 \leq i \leq 3$. Then the points, lines, planes and quadrics in $\mathcal{M}$ are defined to be the subgroups in M conjugate to Q_1, Q_2, Q_3 and Q_8, respectively, with $\mathcal{F} = \{Q_1, Q_2, Q_3, Q_8\}$ being a maximal flag. Let $M_i = N_M(Q_i)$ be the maximal parabolic subgroup corresponding to the flag-transitive action of M on $\mathcal{M}$. Then M_8 is as above while M_1 in the normalizer of a Suzuki type subgroup Q_1 (which we will also denote by σ) and $M_1 \sim 3^{1+12}.2{\cdot}Suz.2$ by 1.1a. The stabilizer of $\mathcal{F}$ in M contains a Sylow 3-subgroup of M. Hence for two elements of $\mathcal{M}$ to be incident it is necessary for their common stabilizer in M to contain a Sylow 3-subgroup. Let $P_i = O_3(M_i)$, P_i^* be the kernel of the action of M_i on the residue of Q_i in $\mathcal{M}$ and $\bar{M}_i = M_i/P_i^*$ for $1 \leq i \leq 4$. It is clear that $Q_i \leq P_i$ and that $Q_8 = P_8 = P_8^*$. For $i = 1$, 2, 3 and 8 by $\mathcal{M}_i$ we denote the set of points, lines, planes and quadrics in $\mathcal{M}$, respectively. For an element α in $\mathcal{M}$ by $\mathcal{M}_i(\alpha)$ we denote the set of elements in $\mathcal{M}_i$ incident to α.

Let Σ be the graph on the Suzuki type subgroups in Q_8 in which two subgroups are adjacent if they are orthogonal with respect to φ. Then Σ is strongly regular with parameters

$$v = 1\,066, \quad k = 336, \quad l = 729, \quad \lambda = 92, \quad \mu = 112.$$

(that is, Σ has $v = 1\,066$ vertices, every vertex has $k = 336$ neighbors and $l = 729$ vertices in distance two, two adjacent vertices have $\lambda = 92$ common neighbors and two vertices of distance two have $\mu = 112$ common neighbors.).

The quotient M_8/Q_8 induces a rank 3 action on Σ, so that if $\sigma \in \Sigma$ then $N_{M_8/Q_8}(\sigma)$ acts transitively on the set $\Sigma_1(\sigma)$ of points adjacent to σ in Σ and on the set $\Sigma_2(\sigma)$ of points at distance 2 from σ.

The next statement follows from standard properties of classical groups.

Lemma 1.3 *Let $L = N_{M_8/Q_8}(\sigma)$ and z be an involution from $O_{3,2}(L)$. Then*

(a) $L \sim 3^6.2{\cdot}U_4(3).2^2$ *and Q_8, as a module for L, has a unique composition series:*

$$1 < \sigma < \langle \Sigma_1(\sigma) \rangle = \sigma^\perp < Q_8;$$

(b) *both $\sigma^\perp/\sigma$ and $O_3(L)$ are isomorphic to the natural orthogonal module for $O^2(L)/O_3(L) \sim 2{\cdot}U_4(3) \sim 2{\cdot}P\Omega_6^-(3)$;*