Smooth, Homogeneous and Isotropic Solutions of Quasilinear Stochastic Partial Differential Equations of McKean-Vlasov Type

Peter Kotelenez
Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106

Abstract

Smooth, homogeneous and isotropic random fields are derived from the distribution of a system of diffusing particles with mean field interaction. The main tools are stochastic ordinary differential equations (SODE's) for the positions of the particles and stochastic partial differential equations (SPDE's) for the mass distribution of the particles. The methods and results used in this paper generalize methods and results of Kotelenez (1995b) from the case of conserved finite mass to the case of infinite mass and the derivation of homogeneous and isotropic fields as solutions of SPDE's.

Contents

AMS Subject Classification (1979): Primary 60H15; Secondary 60H10, 60K35, 60G60, 81G99, 35K55.
Key words: particle systems, stochastic partial differential equations, random fields.

This research was supported by NSF grant DMS 92-11438.

1 Introduction

Let N point particles be distributed over $\mathbf{R}^d$, where d is arbitrary and fixed. The position of the i-th particle at time t will be denoted by $r^i(t)$, which is by assumption in $\mathbf{R}^d$ for all i. We will assume that by some kind of adiabatic elemination we can neglect the momenta (cf. e.g., Kotelenez and Wang, (1994)). Thus we consider the position of the N-particle system at time t as a point in $\mathbf{R}^{dN}$, i.e.,

$$(r^1(t), \ldots, r^N(t)) \in \mathbf{R}^{dN}, \tag{1.1}$$

and a description of its time evolution will be a microscopic model for the particle distribution. Typically one derives from mechanical principles ordinary differential equations (ODE's) or stochastic ordinary differential equations (SODE's) for (1.1) (cf. e.g., Lebowitz and Rubin (1963), Il'in and Khasminskii (1964), Nelson (1972), Kotelenez and Wang (1994) etc.). We will call such equations (describing the position of each particle, usually in dependence on the position of the other particles) *microscopic equations.* Now let the mass of the i-th particle be a_i. Then the empirical mass distribution at time t of the N-particle system is given by

$$\mathcal{X}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i(t))}, \tag{1.2}$$

where δ_r is the point measure, concentrated in $r \in \mathbf{R}^d$. In other words, $\mathcal{X}_N(t)$ is a measure process and for $A \subset \mathbf{R}^d$, $\mathcal{X}_N(t, A) = \sum_{i=1}^N a_i 1_{\{r^i(t) \in A\}}$ describes the mass in A at t irrespective of which particular particles are in A at t, i.e., $\mathcal{X}_N(t)$ reduces information given by (1.1) to the relevant one as far as the mass distribution is concerned. On the other hand, if we agree to call a mass distribution *macroscopic* if the particle structure cannot be seen and stochastic fluctuations are absent then we may call a description of the time evolution of (1.2) a *mezoscopic model*, in particular, if stochastic fluctuations are present in (1.1), which we will always assume. In Kotelenez (1995b) a stochastic partial differential equation (SPDE) for (1.2) was derived, which was called a *mezoscopic equation.* Under physically and mathematically reasonable assumption on (1.1) this SPDE was extended from sums of weighted point measures to measures having densities with respect to the Lebesque measure dr on $\mathbf{R}^d$. Then as a certain parameter, the correlation length of the fluctuation forces in (1.1), tends to 0, the solutions of the SPDE were shown to tend to the solution of an (integro-) partial differential equation (PDE), which was called a *macroscopic equation* for the mass distribution.

This approach was motivated by the author's derivation of a stochastic Navier-Stokes equation for the vorticity of a two-dimensional fluid (Kotelenez (1995a)). The corresponding mezoscopic description $\mathcal{X}_N(t)$ in that case is a sum of point measures multiplied by intensities $a_i \in \mathbf{R}$, i.e., by point measures in the position of point vortices with positive and negative intensities. To include this case as well as other possible signed measure valued cases (cf. Fife (1992)) we will allow the a_i in (1.2) to be positive and negative and just call them positive and negative "masses". So let $a^+ > 0$ be the total "positive" mass of (1.2), $a^- \geq 0$ be the total negative mass and $\mathbf{a} := (a^+, a^-)$. Set

$$\mathbf{M_a} := \{\mu : \mu \text{ is a finite signed Borel measure on } \mathbf{R}^d, \mu^{\pm}(\mathbf{R}^d) = a^{\pm}\},$$

where μ^+ and μ^- is the Jordan decomposition of μ and for any Borel set $A \subset \mathbf{R}^d$ and nonnegative numbers $b^+, b^- \mu^{\pm}(A) = b^{\pm}$ if and only if $\mu^+(A) = b^+$ and $\mu^-(A) = b^-$. First we define a metric on $\mathbf{R}^d$ by

$$\rho(r, q) := (\bar{c}|r - q| \wedge 1), \tag{1.3}$$