STRUCTURE THEOREMS FOR HOMOTOPY PUSHOUTS I: CONTRACTIBLE PUSHOUTS

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Abstract. We define a classifying space for contractible homotopy pushout diagrams and then study its homotopy type.

Introduction

The purpose of this series of papers is to study homotopy pushout squares, i.e., commutative diagrams of spaces of the form

$$\begin{array}{ccc} A & \longrightarrow & C \\ \Big\downarrow & & \Big\downarrow \\ K & \longrightarrow & X \end{array} \tag{0.1}$$

such that the canonical map

$$K \cup_{A\times 0} A \times I \cup_{A\times 1} C \to X$$

is a homotopy equivalence, where the source denotes the double mapping cylinder of the diagram $K \leftarrow A \to C$. In this paper, we will only consider the case $X = \mathrm{pt}$; we may then specify the diagram (0.1) in a compact way as a map

$$A \to K \times C\,.$$

Observe that A homologically behaves as the wedge $K \vee C$. In fact, A splits after one suspension as the wedge of K and C. However, the following elementary example shows that A might not split unstably.

0.2. Example. Let $K = S^m$ and $C = S^m_+ \wedge S^m \simeq S^{2m} \vee S^m$, with $m \geq 1$. Let $A = S^m \times S^m$, with $A \to K$ the projection onto the first factor, and $A \to C$ the map which collapses $S^m \times * \subset S^m \times S^m$ to a point. Then the data describe a contractible homotopy pushout. It is easy to check by means of cohomology rings that A fails to split as $K \vee C$.

Now assume that K and C denote 1-connected, based spaces. Let $\mathfrak{D}(K,C)$ denote the category in which an object is a contractible homotopy pushout diagram of based spaces $A \to K \times C$ such that A is also 1-connected. A morphism from an object $A \to K \times C$ to an object $A' \to K \times C$ is specified by a based weak equivalence $A \to A'$ which commutes with projection to $K \times C$. Let

$$\mathcal{D}(K,C) := |\mathfrak{D}(K,C)|$$

denote the classifying space of $\mathfrak{D}(K,C)$, that is, the geometric realization of its nerve.† Thus, $\mathcal{D}(K,C)$ is a kind of classifying space for contractible homotopy pushouts with vertices K and C. Note that $\mathcal{D}(K,C)$ is equipped with a basepoint given by the 'trivial' diagram $K \vee C \to K \times C$.

Suppose that K is homotopy equivalent to a CW complex of dimension k but not to one of $k-1$. In this instance we write $\dim_{CW} K = k$. Similarly, let $\dim_{CW} C = c$. Let $r = \operatorname{conn} K$ be the connectivity of K, and let $s = \operatorname{conn} C$ denote the connectivity of C. For based spaces Y and Z, let $\{Y, Z\}$ denote the abelian group of homotopy classes of stable maps from Y to Z.

Our first result is

Theorem A. *(1). There is a homotopy equivalence*

$$\mathcal{D}(K,C) \quad \simeq \quad \coprod_{\alpha} BM_\alpha$$

where M_α is a certain group-like topological monoid (defined in 3.1 below), BM_α denotes its classifying space and the disjoint union is indexed over the elements of $\pi_0(\mathcal{D}(K,C))$.

(2). There is a map of pointed sets

$$\pi_0(\mathcal{D}(K,C)) \to \{K \vee C, K \wedge C\}$$

which is surjective if $k, c \le 3\min(r,s)$ and an isomorphism if $k, c \le 3\min(r,s) - 1$.

Without knowledge of the explicit form of the topological monoids M_α, the first part of Theorem A has trivial content: any space decomposes as a disjoint union of its connected components. By choosing a basepoint, each connected component may be written up to weak homotopy equivalence as the classifying space of its associated (Moore) loop space (the latter a topological monoid). The interest in the first part of Theorem A lies in the explicit description of the M_α. In fact, M_α can be defined as the topological monoid of homotopy automorphisms of a choice of basepoint in the α-component of $\mathcal{D}(K,C)$. This description will be used in proving Theorems B and C (see below), and it is for this reason that we state the first part.

†As usual, in order to make sense of geometric realization, we have to replace this category with a suitably 'small' model, by restricting what one means by the term 'space'. For our purposes 'space' will denote a compact ANR embedded in $\mathbb{R}^\infty$. Alternatively, one can work with countable CW complexes. In the end, the model to be used is basically a matter of taste.