On the generalized Fitting group of locally finite, finitary groups

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Abstract

Let G be a locally finite, finitary group and $F^*(G)$ the group generated by the Hirsch-Plotkin radical of G and the components of G. Our main theorem asserts that $C_G(F^*(G)) \leq F^*(G)$.

1 Introduction

The main purpose of this paper is to extend the concept of the generalized Fitting group from finite groups to locally finite, finitary groups. Recall that a group G is called locally finite if every finite subset of G lies in a finite subgroup. G is called finitary if there exist a field K and a faithful KG-module V so that $[V, g]$ is finite dimensional for all $g \in G$. G is quasi-simple if it is perfect, and $G/Z(G)$ is simple. A component of G is a non-trivial quasi-simple, subnormal subgroup of G. The layer $E(G)$ is defined as the group generated by the components of G. It is easy to see that distinct components of a group commute. Thus $E(G)/Z(E(G))$ is semisimple, that is the (restricted) direct product of simple groups.

Assume for the moment that G is finite. Then the Fitting group $F(G)$ is the largest nilpotent normal subgroup of G and the generalized Fitting group $F^*(G)$ is defined as $F(G)E(G)$. A well known theorem (see for example [As]) asserts that $C_G(F^*(G)) \leq F^*(G)$. This means that the group $G/Z(F(G))$ is a subgroup of the automorphism of $F^*(G)$. Thus a finite group G can be well described in terms of $F^*(G)$. As the strucure of $F^*(G)$ is fairly simple compared with the structure of an arbitray finite group, this explains the importance of the generalized Fitting group in the theory of finite groups.

An infinite group G does not necessarily have a largest nilpotent normal subgroup, but it does have a unique maximal locally nilpotent, normal subgroup, the Hirsch-Plotkin radical (see [Ro, page 58]. We denote the Hirsch Plotkin radical by $\mathcal{LN}(G)$ and define the generalized Fitting group $F^*(G)$ of G to be the product $\mathcal{LS}(G)E(G)$. The reader should notice that the generalized Fitting group of a group can be trivial. Indeed this is the case for free groups of rank at least two and even for some infinite, locally finite, groups. Surprisingly the situation is much better for locally finite, finitary groups. Indeed, we will prove:

Theorem A *Let G be a locally finite, finitary group. Then $C_G(F^*(G)) \leq F^*(G)$.*

Similar but different results have been obtained independently by V.V Belyeav [Be, Theorem A] and R.E Phillips [Ph].

As I was kindly reminded by W. Gaschütz, for finite groups the generalized Fitting group can also be described as the set of elements which induce inner automorphism on each chief-factor of the group. 5.11 below implies that same is true for locally finite, finitary groups.

In section 4 we prove that the elements in locally finite, finitary groups have a purely group theoretical property we call *bounded.* (See the end of the introduction for the relevant definitions). In section 5 we prove a structure theorem for arbitray locally finite bounded groups modulo their largest locally solvable normal subgroup. In section 6 we use this structure theorem to prove Theorem A. Finally in section 7 we use the methods and results developed in the earlier sections to show that any locally finite finitary simple group has a Kegel cover which is unipotent by quasi-simple. This last result also has been proved independently in [1, Proposition 4.4] and [Be, Theorem B].

Our results depend on the classification of finite simple groups, but only in form of the Schreier conjecture. That is we assume that the outer automorphism group of every finite simple group is solvable.

We finish the introduction with a list of some of the notation and conventions used throughout this paper.

Let H be group and $\mathcal{P}$ a group theoretical property. H is called locally $\mathcal{P}$ if every finite subset of H lies in a subgroup of H which property $\mathcal{P}$.

F is a finite group.

G is a locally finite group.

$q(F)$ is the number of non-abelian composition factors in a given composition series for F.

Let $X \subseteq Aut(F)$. Then $q(F, X) = \sum_{i=1}^{n} q([F_i/F_{i-1}, X])$, where

$$1 = F_0 < F_1 < \ldots < F_{n-1} \leq F_n < F$$

is a maximal X-invariant subnormal series for F.

Let X a finite subset of G and $n \geq 0$.

$\mathcal{H} = \{H \leq G | H \text{ finite}\}$.

$\mathcal{H}(X) = \{H \in \mathcal{H} | X \subseteq H\}$,

X is n-bounded if there exists $H_X \in \mathcal{H}(X)$ such that $q(F, X) \leq n$ for all $F \in \mathcal{H}(H_X)$ and X is bounded if X is n-bounded for some n.

B_n is the set of n-bounded elements, B_∞ is the set of all bounded elements in G, and G is called bounded if $G = B_\infty$.

$\mathcal{LS}(G)$ is the subgroup of G generated by all the locally solvable normal subgrous of G. Note that $\mathcal{LS}(G)$ itself is locally solvable and so $\mathcal{LS}(G)$ is the largest locally solvable normal subgroup of G.

G^∞ is the group generated by all the perfect finite subgroups of G, i.e G^∞ is the smallest normal subgroup of G such that G/G^∞ is locally solvable.

$S^*(G) = X^\infty$, where $X/\mathcal{LS}(G)$ is group generated by the simple subnormal subgroups of $G/\mathcal{LS}(G)$. Note that $X/\mathcal{LS}(G)$ is the largest semisimple normal subgroup of $G/\mathcal{LS}(G)$.

$S_0^*(G) = 1$ and $S_{n+1}^*(G)$ is defined inductively by $S_{n+1}^*(G)/S_n^*(G) = S^*(G/S_n^*(G))$.

$S_\infty^*(G) = \bigcup_{i=0}^{\infty} S_i^*(G)$.