Defining relations and the algebraic structure of the group SL_2 over integral Hamilton quaternions*

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(dedicated to Professor Robert A.Rankin for his 80th birthday)

In the present paper, we study the group $SL_2(\mathbb{H}_\mathbb{Z})$ of (2×2)-matrices with reduced norm 1 over the noncommutative ring of Hamilton quaternions with integral coefficients. In section 1, we obtain a presentation for $SL_2(\mathbb{H}_\mathbb{Z})$ in terms of generators and defining relations. In section 2, we exhibit a subgroup of finite index of $SL_2(\mathbb{H}_\mathbb{Z})$ with a nonabelian free quotient of rank 3. It is well known that the group $SL_2(\mathbb{Z})$ has the same property. In fact, the commutator subgroup of $SL_2(\mathbb{Z})$ is a free group of rank 2 and has index 12. The normal subgroup generated in $SL_2(\mathbb{Z})$ by the element

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{1}$$

coincides with $SL_2(\mathbb{Z})$ if $m = 1$, is of finite index if $1 < m \leq 5$ and has infinite index otherwise. In $SL_2(\mathbb{H}_\mathbb{Z})$, the normal closure of the element (1) also coincides with the whole group if $m = 1$. But the normal closure of (1) in $SL_2(\mathbb{H}_\mathbb{Z})$ turns out to be of infinite index if m = 2.

We have tried to keep the exposition of the paper as elementary as possible.

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1 Defining relations for $SL_2(\mathbb{H}_\mathbb{Z})$

Since the ring $\mathbb{H}_\mathbb{Z}$ is Euclidean it follows easily that $SL_2(\mathbb{H}_\mathbb{Z})$ is generated by the elementary unipotent matrices:

$$\begin{aligned} q_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \\ q_3 &= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad q_4 = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \\ q_5 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad q_6 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \\ q_7 &= \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}, \quad q_8 = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}. \end{aligned} \tag{2}$$

It is easy to check that the generators (2) can be expressed in terms of the matrices

$$\begin{aligned} a &= \begin{pmatrix} k & 0 \\ 0 & -j \end{pmatrix}, \quad b = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}, \\ c &= \begin{pmatrix} k & 1 \\ 0 & -k \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{3}$$

as follows:

$$\begin{aligned} q_1 &= dbdacdadb, & q_2 &= bq_1b^{-1}, \\ q_3 &= aq_1{}^{-1}a^{-1}, & q_4 &= baq_1{}^{-1}a^{-1}b^{-1}, \\ q_5 &= dq_1{}^{-1}d^{-1}, & q_6 &= dq_2{}^{-1}d^{-1}, \\ q_7 &= dq_3{}^{-1}d^{-1}, & q_8 &= dq_4{}^{-1}d^{-1}. \end{aligned} \tag{4}$$

So we shall use the elements (3) as generators for $SL_2(\mathbb{H}_\mathbb{Z})$. Using the following well known representation of the quaternions:

$$\begin{aligned} i &\to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ k &\to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad 1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

we represent any element

$$x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}_\mathbb{Q} \tag{5}$$

as a (2×2)-matrix over $\mathbb{Q}[i]$:

$$\begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \quad (x_j \in \mathbb{Q}).$$