A Visualization of the Isometry Group Action on the Fomenko–Matveev–Weeks Manifold

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Abstract

The smallest known three-dimensional closed orientable hyperbolic manifold $\mathcal{M}_1$, which volume is equal to $0.94\ldots$, was obtained independently by A. Fomenko and S. Matveev and by J. Weeks. It is known, that the isometry group of the manifold $\mathcal{M}_1$ is isomorphic to the dihedral group $\mathbb{D}_6$ of order 12. We study the lattice of the action of the isometry group on the manifold $\mathcal{M}_1$ and obtain all orbifolds which arised as the quotient spaces.

1. Introduction

In the present paper we study the properties of the smallest known three-dimensional closed orientable hyperbolic manifold $\mathcal{M}_1$, which was constructed independently by A. Fomenko and S. Matveev [7] and by J. Weeks [13].

A three-dimensional hyperbolic manifold can be defined as a quotient space $M = \mathbb{H}^3/\Gamma$, where Γ is a discrete group of isometries of the three-dimensional Lobachevsky space $\mathbb{H}^3$, acting without fixed points. The concept of the volume in $\mathbb{H}^3$ is carried over naturally to M. Further we will consider three-dimensional orientable hyperbolic manifolds of finite volume.

The structure of the set of volumes of three-dimensional hyperbolic manifolds was described by W. Thurston and T. Jørgensen. According to the Thurston–Jørgensen theorem [12], volumes of three-dimensional hyperbolic manifolds form a well-ordered

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non-discrete subset of the real line of the order type ω^ω. In particular, there exists a closed manifold with the smallest volume.

The first manifold which pretended to be the smallest closed manifold was constructed by W. Thurston [12]. This manifold was obtained by $(5,-1)$ Dehn surgery on the figure-eight knot, and its volume is equal to $0.98\ldots$. But several years later the another pretender appeared.

In [13] J. Weeks calculated volumes of hyperbolic manifolds, obtained by Dehn surgeries on hyperbolic knots and links of the small order. It was possible due to his nice computer program *SnapPea* [11]. As the result of the calculations, the closed hyperbolic manifold $\mathcal{M}_1$ with the volume equals to $0.94\ldots$ was founded. The manifold $\mathcal{M}_1$ was obtained by Dehn surgeries with parameters $(5,-2)$ and $(5,-1)$ on both components of the Whitehead link. At the same time, numerous computer calculations of volumes were given by A. Fomenko and S. Matveev [7]. The manifold $\mathcal{M}_1$ also was appeared in their list (but it was obtained by another way). Further the manifold $\mathcal{M}_1$ will be referred to as the *Fomenko–Matveev–Weeks manifold*.

Moreover, A. Fomenko and S. Matveev [7] conjectured the structure of the initial segment of the set of volumes of three-dimensional hyperbolic manifolds. In [5] C. Hodgson and J. Weeks refined the smallest manifolds computing a lot of volumes by SnapPea program.

In [9] E. Molnar constructed the fundamental polyhedron in $\mathbb{H}^3$ for the fundamental group of the manifold $\mathcal{M}_1$ and remarked, that the isometry group of this manifold contains the dihedral group $\mathbb{D}_6$ of order 12. Using computer calculation, it was shown in [5], that the isometry group of $\mathcal{M}_1$ has exactly 12 elements.

The aim of the present paper is to describe the lattice of the action of the isometry group $\mathrm{Isom}(\mathcal{M}_1)$ on the manifold $\mathcal{M}_1$. We will obtain all orbifolds which arise as quotient spaces of $\mathcal{M}_1$ by the action of the subgroups of $\mathrm{Isom}(\mathcal{M}_1)$. In particular, we will describe the manifold $\mathcal{M}_1$ as the two-fold covering of the 3-sphere, branched over the knot 9_{49} and as the cyclic three-fold covering of the 3-sphere, branched over the two-bridge knot 5_2.

2. The lattice of the isometry group's action

Let us consider the fundamental group $\Gamma = \pi_1(\mathcal{M}_1)$ of the Fomenko-Matveev-Weeks manifold $\mathcal{M}_1$. According to [9], the group Γ has the following presentation:

$$\Gamma = \langle\, a, b \mid abab^2a^{-2}b^2 = baba^2b^{-2}a^2 = 1 \,\rangle. \tag{1}$$

It was observed by E. Molnar [9], that the group Γ has the following three automorphisms

$$\left\{ \begin{array}{l} a \to a^{-1}, \\ b \to b^{-1}, \end{array} \right. \qquad \left\{ \begin{array}{l} a \to b, \\ b \to a, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} a \to a, \\ b \to a^{-1}b^{-1}. \end{array} \right. \tag{2}$$

By the Mostow rigidity theorem, there exist isometries r, s and t of the Lobachevsky space $\mathbb{H}^3$ such that