A short proof of Dirac's theorem on the number of edges in chromatically critical graphs

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Abstract

A short proof is presented for a theorem of G. A. Dirac (1974) giving a lower bound for the number of edges of a k-critical graph on a given number of vertices and describing all non-complete k-critical graphs with minimum edge excess.

1 Introduction

All graphs considered in this note are finite, undirected and have neither loops nor multiple edges. For a graph G, $V(G)$ is its vertex set and $E(G)$ is its edge set; $D_G(x)$ denotes the degree (valency) of $x \in V(G)$. Througout, k denotes an integer not smaller than 4. G is called k-critical if the chromatic number of G is k and each proper subgraph of G is $(k-1)$-colourable. Let $\mathcal{C}(k)$ denote the set of k-critical graphs. Clearly,

$$\min\{d_G(x) \mid x \in V(G),\, G \in \mathcal{C}(k)\} = k-1.$$

This definition gives rise to the definition of the *degree excess* $\epsilon(G)$ for $G \in \mathcal{C}(k)$:

$$\epsilon(G) := \sum_{x \in V(G)} (d_G(x) - (k-1)) = 2|E(G)| - (k-1)|V(G)|.$$

Let K_m denote the complete graph on m vertices. Making use of Brooks' theorem, we immediately obtain

$$\epsilon(G) = 0 \quad \text{if and only if} \quad G \simeq K_k.$$

Let $\mathcal{C}^0(k) = C(k) \setminus K_k$. G. A. Dirac [2] proved

*Research of this author was supported in part by the grant 96-01-01614 of the Russian Foundation for Fundamental Research, by the Network DIMANET of the European Union and SFB 343 "Diskrete Strukturen in der Mathematik".

Theorem 1 *$\epsilon(G) \geq k-3$ for every $G \in \mathcal{C}^0(k)$.*

Shorter proofs were given by Kronk and Mitchem [5] and Weinstein [7]. The bound is attained by all graphs in the family $\mathcal{D}(k)$ that is defined as follows. Each member D of $\mathcal{D}(k)$ has $2k-1$ vertices partitioned into three subsets: V', V'' and $\{v_1, v_2\}$ where $|V'| = k-2$, $|V''| = k-1$, the subgraphs of D induced by $V' \cup \{v_1\}$, $V' \cup \{v_2\}$, and V'' are complete graphs, each vertex in V'' is adjacent to exactly one of v_1 and v_2, each of v_1 and v_2 is adjacent to some vertex in V'', and there are no other edges.

In 1974, Dirac [3] extended Theorem 1 as follows.

Theorem 2 *For every $G \in \mathcal{C}^0(k) \setminus \mathcal{D}(k)$,*

$$\epsilon(G) \geq \begin{cases} 2, & \text{if } k = 4; \\ k-1, & \text{if } k > 4. \end{cases}$$

The proof is rather long. Mitchem [6] gave a shorter proof based on Kempe chains. The aim of the present note is to give a shorter proof of Theorem 2 based on a different idea.

2 Proof of Theorem 2

Lemma 1 (Dirac [1]) *Each edge cut in a k-critical graph has at least $k-1$ edges.* $\square$

For a simple proof, see [4].

Let $G = (V, E) \in \mathcal{C}^0(k)$ and $x, v, w \in V$ be such that $xv, xw \in E$ and $vw \notin E$. We obtain $H = H(G; x, v, w)$ from G by amalgamating v and w into a new vertex $v * w$ (replacing double edges by single ones). Since the chromatic number of H is at least k, H contains a k-critical subgraph. The set of all k-critical subgraphs of H is denoted by $\mathcal{R}(G; x, v, w)$. Graph $H - v * w$ being a proper subgraph of G, its chromatic number is smaller than k, therefore

$$v * w \in V(G^*) \text{ for every } G^* \in \mathcal{R}(G; x, v, w). \tag{1}$$

If $d_G(x) = k-1$, then $d_H(x) = k-2$ thus

$$d_G(x) = k-1 \text{ implies } x \notin V(G^*) \text{ for every } G^* \in \mathcal{R}(G; x, v, w). \tag{2}$$

From now on, assume that $G = (V, E)$ is a counterexample to the theorem with the minimum number n of vertices. We denote by K_m^- the graph obtained from K_m by deleting one edge. For disjoint $A, B \subset V$, $E_G(A, B)$ is the set of edges in G joining A with B.