On the Storage Capacity of Hopfield Models with Spatially Correlated Patterns

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Abstract

We analyze the storage capacity of the Hopfield model with spatially correlated patterns ξ_i^ν (i.e. the patterns ξ^ν themselves are independent but have correlated components ξ_i^ν). We show that the standard Hopfield model of neural networks with N neurons can store $\frac{N}{\gamma \log N}$ or αN semantically correlated patterns (depending on which notion of storage is used), provided that the correlation comes from a homogeneous Markov chain. The quantity γ increases with an increasing correlation of the patterns, while α decreases with large correlations. This shows that the standard Hopfield can store an increasing number of spatially correlated patterns. The proofs use adaption of the methods developed for the case of i.i.d. or semantically correlated patterns to the situation of spatial correlation together with an estimate of the determinant of a symmetric matrix with trace zero.

1 Introduction

The study of neural networks has been a major topic of research in mathematics, physics, and computer science during the past 15 years.

Basically a model of a neural network consists of a labeled and possibly oriented graph $G = (V, E)$ together with a set S with $card(S) \geq 2$ to describe the set of neurons (by V), the synapses connecting these neurons (by E), and the activity of each of the neurons (by a variable $\sigma_i \in S$ for $i \in V$). Moreover, the information to be stored is encoded in so-called patterns $\xi^\mu, \mu = 1, \ldots, M(N)$ each of the ξ^μ itself being a sequence of $\xi_i^\mu \in S, i = 1, \ldots, N$. Finally, to make the neural net capable of adapting to different sequences of patterns, we have to introduce a set of variables $J_{ij}, ij \in E$ called the synaptic efficacies and describing the strength of interaction between the neurons at sites i and j. It is commonly assumed that the variable J_{ij} is measurable with respect to the set $\{\xi_i^\mu, \xi_j^\mu, \mu = 1, \ldots, M(N)\}$ (so-called *locality* of the weights J_{ij}). The basic idea now behind this setup is to define a Hamiltonian $H_N(\sigma)$ on S^N, such that the Monte-Carlo dynamics governed by this Hamiltonian eventually converges to one of $M(N)$ states close to the original patterns.

One of the classical and best understood examples of a neural network is the so-called Hopfield model [Ho82]. Although originally introduced by Pastur and Figotin [FP77] as a simplified model of a spin–glass, this model achieved much of its interest by its reinterpretation as an auto-associative memory by Hopfield and may therefore by right be called the Hopfield model. Here the graph G is the complete graph K_N on N vertices, $S = \{-1, +1\}$, corresponding to that a neuron may be either switched "on" or "off" and the weights J_{ij} are given by "Hebb's learning rule", i.e. the formula

$$J_{ij} := \frac{1}{N} \sum_{\mu=1}^{M(N)} \xi_i^\mu \xi_j^\mu.$$

One of the most important advances due to Hopfield [Ho82] has been to understand that this setup corresponds to a Hamiltonian H_N given by

$$H_N(\sigma) := -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j = -\frac{1}{2N} \sum_{i,j=1}^{N} \sum_{\mu=1}^{M(N)} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j. \tag{1}$$

Note that (1) may be rewritten in the very convenient form

$$H_N(\sigma) = -\frac{N}{2} \sum_{\mu=1}^{M(N)} (m_N^\mu(\sigma))^2 = -\frac{N}{2} ||m_N(\sigma)||_2^2 \tag{2}$$

where $||m_N(\sigma)||_2^2$ is the l_2-norm of the so-called overlap vector

$$m_N(\sigma) := (m_N^\mu(\sigma))_{\mu=1,\ldots,M(N)} = (\frac{1}{N} \sum_{i=1}^{N} \sigma_i \xi_i^\mu)_{\mu=1,\ldots,M(N)}.$$

Note that $m_N(\sigma)$ may be regarded as an index for how much a configuration is correlated to one of the given patterns (large absolute values of a component of $m_N(\sigma)$