Exceptional Sequences in Hall Algebras and Quantum Groups*

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Abstract. Let Λ be a finite dimensional hereditary algebra over a finite field k, $\mathcal{H}(\Lambda)$ and $\mathcal{C}(\Lambda)$ be respectively the Hall algebra and the composition algebra of Λ, $\mathcal{P}$ be the isomorphism classes of finite dimensional Λ-modules and I the isomorphism classes of simple Λ-modules. We define δ_α and ${}_\alpha\delta$, $\alpha \in \mathcal{P}$, to be the right and left derivations of $\mathcal{H}(\Lambda)$ respectively. By using these derivations and the action of the braid group on the set of exceptional sequences of Λ-mod , we provide an effective algorithm of calculating root vectors. This means that we get an inductive method to express u_λ as the combinations of elements u_i in the Hall algebra, where $i \in I$ and $\lambda \in \mathcal{P}$ is any exceptional Λ-module. Because of the canonical isomorphism between the Drinfeld-Jimbo quantum group and the generic composition algebra, our algorithm is applicable directly to quantum groups.

1 Introduction

1.1 Let $\mathbf{g}$ be a semisimple Lie algebra over the complex field $\mathbf{C}$, $\mathbf{U}(\mathbf{g})$ be the universal enveloping algebra of $\mathbf{g}$. It is well-known that $\mathbf{g}$ has a root space decomposition

$$\mathbf{g} = \mathbf{g_0} \bigoplus \coprod_{\alpha \in \mathbf{\Phi}} \mathbf{g}_\alpha$$

with the root system Φ. Moreover , for any $\alpha \in \Phi$, $\dim_{\mathcal{C}} \mathbf{g}_\alpha = \mathbf{1}$. Choose a complete set of simple roots in Φ and denote it by I, then the root system Φ is divided into two parts: $\Phi = \Phi^+ \bigcup \Phi^-$, where Φ^+ and Φ^- are positive and negative roots respectively. Therefore $\mathbf{g}$ has a triangular decomposition

$$\mathbf{g} = \mathbf{g}^- \bigoplus \mathbf{g_0} \bigoplus \mathbf{g}^+.$$

For any $\alpha \in \Phi^+$, choose nonzero $x_\alpha \in \mathbf{g}_\alpha$ such that $\{x_\alpha \in \mathbf{g}_\alpha | \alpha \in \mathbf{\Phi}^+\}$ constitute a basis of $\mathbf{g}^+$. Moreover, it is possible to choose $x_\alpha \in \mathbf{g}_\alpha(\alpha \in \mathbf{\Phi}^+)$ such that $\{x_\alpha \in \mathbf{g}_\alpha | \alpha \in \mathbf{\Phi}^+\}$ is a Chevalley basis of $\mathbf{g}^+$. The vectors $\{x_\alpha \in \mathbf{g}_\alpha | \alpha \in \mathbf{\Phi}^+\}$ are called *root vectors.* Of course $\mathbf{g} \subset \mathbf{U}(\mathbf{g})$ as Lie subalgebra and $\mathbf{U}(\mathbf{g})$ has the triangular decomposition

$$\mathbf{U}(\mathbf{g}) = \mathbf{U}^- \otimes \mathbf{U^0} \otimes \mathbf{U}^+$$

which satisfies $\mathbf{g}^+ \subset \mathbf{U}^+, \mathbf{g}^- \subset \mathbf{U}^-$ and $\mathbf{g_0} \subset \mathbf{U^0}$.

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It is well-known that the semisimple Lie algebra $\mathbf{g}$ itself can not be quantized, but according to Drinfeld and Jimbo, the universal enveloping algebra $\mathbf{U}(\mathbf{g})$ of $\mathbf{g}$ admits a nontrivial quantization with q as a parameter, that is the so-called quantum group or the quantum enveloping algebra $\mathbf{U_q}(\mathbf{g})$ of $\mathbf{g}$. When $q \neq 1$, there is no longer $\mathbf{g} \subset \mathbf{U_q}(\mathbf{g})$. However according to Lusztig [L], there is an action of the braid group on $\mathbf{U_q}(\mathbf{g})$. It is still possible to obtain a family of linearly independent elements of $\mathbf{U_q^+}(\mathbf{g})$ by applying the braid group action in an admissible order on the generators of $\mathbf{U_q^+}(\mathbf{g})$. Those elements obtained are degenerated into a basis of $\mathbf{g}$ along $q \to 1$. So we also call those elements the *root vectors* of $\mathbf{U_q^+}(\mathbf{g})$.

1.2 Given any Dynkin diagram Δ of type $A_n, B_n, C_n, E_i (i = 6, 7, 8), F_4$ and G_2 and any finite field k, there exists a finite dimensional hereditary k-algebra Λ corresponding to Δ. Let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional Λ-modules. The Hall algebra $\mathcal{H} = \mathcal{H}(\Lambda)$ is by definition the free $\mathbf{Z}[v, v^{-1}]$-module with the basis $\{u_\alpha | \alpha \in \mathcal{P}\}$ where $v^2 = q$ and $q = |k|$; the multiplication is given by

$$u_\alpha u_\beta = v^{\langle \alpha, \beta \rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha\beta}^\lambda u_\lambda$$

for all $\alpha, \beta \in \mathcal{P}$, where

$$\langle \alpha, \beta \rangle = \dim_k \operatorname{Hom}_\Lambda(V_\alpha, V_\beta) - \dim_k \operatorname{Ext}_\Lambda(V_\alpha, V_\beta)$$

with $V_\alpha \in \alpha, V_\beta \in \beta$ and $g_{\alpha\beta}^\lambda$ is the number of submodules X of V_λ such that V_λ / X and X lie in the isomorphism classes α and β respectively. Let $I \subset \mathcal{P}$ be the set of isomorphism classes of simple Λ-modules. Ringel [R2,R3] has proved there exists a canonical isomorphism η between the Lusztig's quantum group f and $\mathcal{H}(\Lambda)$ such that $\eta(\theta_i) = u_i, i \in I$, if they enjoy a common Dynkin diagram. Furthermore, the generic form $\mathcal{H}(\Delta)$ of $\mathcal{H}(\Lambda)$ is canonically isomorphic to the Drinfeld-Jimbo quantum group $\mathbf{U_q^+}(\mathbf{g})$ if the both enjoy a common Dynkin diagram. Under the canonical isomorphism, Ringel has shown that the set $\{u_\lambda | \lambda \in \mathcal{P}, V_\lambda \text{ indecomposable}\}$ just provide a complete set of root vectors of $\mathbf{U_q^+}(\mathbf{g})$.

1.3 One may ask the question: how to decompose the root vectors into the combinations of the generators of $\mathbf{U_q^+}(\mathbf{g})$. If we apply the Lusztig's braid group operations on $\mathbf{U_q^+}(\mathbf{g})$ to deal with this question, a trouble appears immediately. Namely we can not expect that the computation is undertaken in the interior of $\mathbf{U_q^+}(\mathbf{g})$. Oppositely, it often goes across into the negative part $\mathbf{U_q^-}(\mathbf{g})$, although the final result of the calculation lies in $\mathbf{U_q^+}(\mathbf{g})$. Ringel has pointed out in [R4] that the braid group action on the exceptional sequences of Λ-mod in the sense of Crawley-Boevey [CB] may provide an inductive constructure of root vectors. We will give a refinement of his idea in the present paper. Namely we will give an accurate recursive formula to express u_λ as the combinations of $u_i (i \in I)$ for any exceptional module $\lambda \in \mathcal{P}$. According to our formula, the whole calculation is going on in the interior of $\mathbf{U_q^+}(\mathbf{g})$, moreover, it does not depend on the constructure of the exceptional sequences.

It is well-known that any indecomposable Λ-module is exceptional if Λ is finite representation type. So our result is suitable for the quantum enveloping algebra of semisimple Lie algebra. In fact our research is made in the Hall algebras and the composition algebras of any type. Because of the fundamental theorem of Green and Ringel, namely, there exists a canonical isomorphism