Tilting decompositions of semisimple complex Lie algebras *

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Abstract. Let B be a tilted algebra of Dynkin type Δ. Let $\mathbf{g}$ be the semisimple complex Lie algebra of type Δ. It is proved in this note that the Lie subalgebra of the degenerate Hall algebra $\mathcal{H}_1(B)$ determined by all indecomposable B-modules, can be embedded in the Lie algebra $\mathbf{g}$ in a natural way. This leads to a decomposition of $\mathbf{g}$ analogous to the triangular decomposition of $\mathbf{g}$.

Key Words: tilted algebras, Hall algebras, derived categories, Lie algebras.

0. Introduction

Let A be an associative hereditary algebra over some finite field k of Dynkin type Δ. Let $\mathbf{g}$ be the semisimple complex Lie algebra of type Δ with triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$. C.M.Ringel in [1] showed that A, as a directed algebra, has Hall polynomials. So he could make the free abelian group $\oplus_{[M]}\mathbf{Z}\mathbf{u}_{[\mathbf{M}]}$, indexed by the set of isomorphism classes $[M]$ of all finitely generated A-modules M, into an associative ring by defining:

$$u_{[M]}u_{[N]} = \sum_{[L]} \phi^L_{MN}(1)u_{[L]},$$

for any $M, N \in$mod$A$, where ϕ^L_{MN} are the Hall polynomials corresponding to the filtration sets $\{X \subset L : X \simeq N \text{ and } L/X \simeq M\}$; this ring is called the degenerate Hall algebra of A and denoted by $\mathcal{H}_1(A)$. Let $K(\text{mod}A)$ be the Grothendieck group of mod$A$, the category of all finitely generated (right) A-modules, modulo split exact sequences. Then $K(\text{mod}A)$ can be identified with the subgroup $\oplus_{[X]}\mathbf{Z}u_{[X]}$ of $\mathcal{H}_1(A)$, indexed by the set of isomorphism classes $[X]$ of all indecomposable modules in mod$A$. He showed that $K(\text{mod}A)$ is a Lie subalgebra of $\mathcal{H}_1(A)$, and furthermore he has proved in [2] that $K(\text{mod}A) \otimes_{\mathbf{Z}} \mathbf{C}$ is isomorphic to $\mathbf{n}_+$ as Lie algebras. So we can think that the triangular decomposition of $\mathbf{g}$ is one corresponding to a hereditary partition.

Let B be a tilted algebra of type Δ. Similarly, we have the degenerate Hall algebra $\mathcal{H}_1(B)$ of B and $K(\text{mod}B)$ is a Lie subalgebra of $\mathcal{H}_1(B)$. In this note, we will point out that $\mathbf{g}$ has another decomposition $\mathbf{g} = \mathbf{a}_- \oplus \mathbf{b}_- \oplus \mathbf{h} \oplus \mathbf{b}_+ \oplus \mathbf{a}_+$ with $\mathbf{a}_+ \simeq K(\text{mod}B) \otimes_{\mathbf{Z}} \mathbf{C}$ as Lie algebras, where $\mathbf{h}$ is the Cartan subalgebra. We call it as the tilting decomposition of $\mathbf{g}$.

*The research was supported partially by the fellowship from the Alexander von Humboldt Foundation (Peng) and by Deutsche Forschungsgemeinschaft and SFB(343) (Xiao), and also supported partially by the National Natural Science Foundation of China.

1. A Realization of g

As mentioned, Ringel has given a realization of the positive part of a semisimple complex Lie algebra **g** via the module category of a hereditary algebra. This stimulated the authors to consider how to realize the whole Lie algebra **g** using Hall algebra methods. In [4], we have done this successfully via the derived category of a hereditary algebra. Now in this section we recall this in some details and fix notations, which will be used in the next section to establish and prove the tilting decomposition of **g**.

As before, let A be a hereditary algebra over a finite field k of Dynkin type Δ, and let **g** be the semisimple complex Lie algebra of type Δ.

Let $D^b(A)$ be the derived category of mod$A$, which is a triangulated category with the translation functor T. Denote by $\mathcal{R}$ the orbit category $D^b(A)/T^2$, called the root category of mod$A$(or, Simply, of A), and let $F : D^b(A) \longrightarrow \mathcal{R}$ be the canonical covering functor. The translation functor T under F induces a translation functor of $\mathcal{R}$, still denoted by T, and so $T^2 = 1$ in $\mathcal{R}$. The root category $\mathcal{R}$ with T still is a triangulated category (see Appendix of [4]) and F is exact.

For $X, Y, L \in \mathcal{R}$, let

$$W(X,Y;L) = \{(f,g,h) : X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX \quad \text{is a triangle}\};$$

for $(f,g,h) \in W(X,Y;L)$, denote

$$(f,g,h)^\wedge = \{(\alpha f, g\beta^{-1}, \beta h (T\alpha)^{-1}) : \alpha \in \mathrm{AutX}, \beta \in \mathrm{AutY}\},$$

clearly $(f,g,h)^\wedge \subset W(X,Y;L)$; let

$$V(X,Y;L) = \{(f,g,h)^\wedge : (f,g,h) \in W(X,Y;L)\}.$$

Let E be a field extension of k. For any k-space V, denote $V^E = V \otimes_k E$. So A^E naturally becomes an E-algebra and M^E an A^E-module for any A-module M. Hence for any object $N \in D^b(A)$, a bounded complex of A-modules, N^E makes sense in a natural way, and so does X^E for any $X \in \mathcal{R}$. We call E conservative (for $\mathcal{R}$) if $(\mathrm{End}X/\mathrm{rad}(\mathrm{End}X))^E$ is a field for any indecomposable object $X \in \mathcal{R}$. Note that there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{R}$. Then it is easy to see that there are infinitely many isomorphism classes of conservative finite field extensions of k.

For $M, N, L \in \mathcal{R}$, an integral polynomial $\phi^L_{NM}(q) \in \mathbf{Z}[q]$ is called a Hall polynomial of $\mathcal{R}$ if

$$\phi^L_{NM}(|E|) = |V(M^E, N^E; L^E)|$$

for any conservative finite field extension E of k.

Proposition 1.1. $\mathcal{R}$ has the Hall polynomials ϕ^L_{MN} and ϕ^L_{NM} for any $M, N, L \in \mathcal{R}$ with N indecomposable.

Now denote by $\mathcal{H}$ the abelian group with generators $\{H_{[M]}\}_{[M]}$, indexed by the set of isomorphism classes $[M]$ of all objects M of $\mathcal{R}$, and with relations

$$H_{[M \oplus N]} - H_{[M]} - H_{[N]}$$