ON THE EQUIVARIANT HOMOTOPY OF FREE ABELIAN GROUPS ON G-SPACES AND G-SPECTRA

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1. Introduction

The classical Dold-Thom theorem [DT56] asserts that the singular homology of a space X can be obtained as the homotopy groups $\pi_j(\mathrm{AG}\,(X))$ of the topological free abelian group $\mathrm{AG}\,(X)$ on X. One of the main purposes of this paper is to provide *unstable and stable equivariant* versions of this result.

We obtain results both in the unstable and in the stable equivariant worlds, for a fixed finite group G. In the unstable context, we start with the simple observation that for a G-space X, $\mathrm{AG}\,(X)$ gives rise to a Mackey functor F_X with values in the category of topological abelian groups. It follows that the composition of any homotopy functor with

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F_X yields ordinary Mackey functors. For example one could use any generalized homology or cohomology theory H_* and obtain Mackey functors $H_* \circ F_X$, etc.

Our first and preliminary result is the following Dold-Thom type theorem.

Theorem 4.5. *The homotopy groups of F_X compute the Bredon homology of X with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.*

The main technical tool here is the fact, proven in Theorem 2.7, that *for a G-$\mathcal{CW}$-pair (X, A) the projection $AG(X) \to AG(X/A)$ is a locally trivial, numerable $(G, \alpha, AG(A))$-bundle.* It is worthwhile to mention that the restriction and transfer maps for Bredon homology have a very simple description in terms of the topological Mackey functor structure of $\mathrm{AG}(X)$ itself.

It is well-known that the Bredon homotopy groups of $\mathcal{Z}(X)$ do not completely determine its equivariant homotopy type, contrary to the non-equivariant situation. In fact, Nishida [Nis84] proves that the homotopy category of topological $k[G]$-modules, for a ring k, is equivalent to the derived category of complexes of $k[G]$-modules, and this result is also called an equivariant Dold-Thom theorem. See also [Tri83].

In the stable context our contructions become more subtle, although the results are equally natural. We assume very little knowledge from the reader about the equivariant stable category developed in [LMS85], and we provide essentially all basic definitions and detailed references to all key results used. We first apply the functor $\mathrm{AG}(-)$ spacewise to G-prespectra E and, after suitably defining the structural maps, one then obtains a G-prespectrum $\mathrm{AG}(E)$. This assignment is functorial and reasonably well-behaved with respect to basic operations on prespectra. One then uses the machinery of [LMS85] to pass from prespectra to spectra and denote by $\mathbb{Z}(E)$ the resulting G-spectrum associated to E.

The first stable result obtained identifies $\mathbb{Z}(\mathbb{S})$, where $\mathbb{S}$ is the sphere spectrum:

Theorem 6.4. *Given a finite group G, let A_G denote its Burnside ring Mackey functor. Then one has a natural equivalence of spectra $\mathbb{Z}(\mathbb{S}) \cong K(A_G, 0)$, where $K(A_G, 0)$ denotes an equivariant Eilenberg-MacLane spectrum.*

We then proceed to identify $\mathbb{Z}(E)$ in the stable homotopy category $hG\mathcal{S}$ of G-spectra for an arbitrary G-spectrum E. We construct a natural morphism of spectra $R_E : \mathbb{Z}(\mathbb{S}) \wedge E \to \mathbb{Z}(E)$ which satisfies the following property.

Theorem 6.7. *The natural maps $R_E : \mathbb{Z}(\mathbb{S}) \wedge E \to \mathbb{Z}(E)$ are weak homotopy equivalences.*

One immediately obtains several consequences.

Corollary 6.8 [Stable Dold-Thom theorem]. *For any G-spectrum E, the homotopy groups of $\mathbb{Z}(E)$ compute the ordinary $RO(G)$-graded homology of E with coefficients on the Burnside ring Mackey functor, as defined in* [LMM81]. *In other words, for a G-spectrum E and subgroup $H < G$, one has $\pi_*^H(\mathbb{Z}(E)) \cong H_*^G(E, \mathcal{A}_G)[G/H]$.*