More Tight Monomials in Quantized Enveloping Algebras

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1 Introduction

Let $U = U_q(\mathbf{g})$ be the quantum group associated to a semisimple Lie algebra $\mathbf{g}$ of rank n, by Drinfel'd and Jimbo (see [6] and [7]). The negative part U^- of U has a canonical basis $\mathbf{B}$ (see Kashiwara [8] and Lusztig [11, 14.4.6]) with some nice properties. To explicitly calculate the elements of $\mathbf{B}$ is a hard problem; it is solved completely only in types A_1 and A_2 (see [10, 3.4]). In [12], Lusztig describes a method to calculate elements of $\mathbf{B}$, in terms of so-called tight monomials. A monomial in U^+ in this sense is a product of divided powers of the generators of U^+ (see section 2), and it is said to be tight (respectively, semi-tight) if it belongs to $\mathbf{B}$ (respectively, is a nonnegative integer linear combination of elements of $\mathbf{B}$). Let W be the Weyl group corresponding to $\mathbf{g}$, with Coxeter generators $s_1, s_2, \ldots, s_n$, and let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ be a reduced expression for the longest word in W. Lusztig shows (amongst other things) that, given certain linear conditions on $a_1, a_2, \ldots, a_m \in \mathbb{N}$, the monomial

$$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_m}^{(a_m)} \tag{1}$$

is tight (respectively semi-tight) provided a certain quadratic form, dependent on the reduced expression for w_0 chosen and an orientation of the Dynkin diagram, satisfies a certain positivity condition (respectively, nonnegativity condition). We show that (for a certain orientation) this condition is always satisfied in type A_4 (as opposed to the claim in [12]). We also demonstrate Lusztig's comment that things get more complicated in cases with higher rank in type A. In particular, we exhibit a reduced expression for w_0 in type A_r (for any $r \geq 7$) with a quadratic form that does not even satisfy the condition for semi-tightness, for any orientation of the Dynkin diagram.

2 Preliminaries

We use the treatment in [11, §§1-3]. Let $\mathbf{g}$ be a semisimple Lie algebra, with root system Φ, simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and Killing form $(\ ,\ )$. Let $h_1, h_2, \ldots, h_n$ be a basis for a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, satisfying $(h_i, h) = \alpha_i^*(h)$ for all h in $\mathbf{h}$ and all $i \in I = \{1, 2, \ldots, n\}$. Let Y be the $\mathbb{Z}$-lattice spanned by $h_1, h_2, \ldots, h_n$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental weights of $\mathbf{g}$, defined by $\omega_i(h_j) = \delta_{ij}$, and let X be the $\mathbb{Z}$-lattice spanned by them (the weight lattice). Let d be the minimal positive integer so that $d(\alpha_i, \alpha_j)$ is always an integer and $d(\alpha_i, \alpha_i)$ is always even. If the highest common factor of the $d(\alpha_i, \alpha_j)$ and the $\frac{1}{2}d(\alpha_i, \alpha_i)$ is not 1, then replace d by d divided by this highest common factor. We then define $i \cdot j$ to be $d(\alpha_i, \alpha_j)$ for each $i, j \in I$, so $(I, \cdot)$ is a Cartan datum as in [11, 1.1.1]. For $\mu \in Y$ and $\lambda \in X$, define $\langle \mu, \lambda \rangle$ to be $\lambda(\mu)$. Define an embedding of I into Y by $i \mapsto h_i$ and into X by $i \mapsto \alpha_i$ for all $i \in I$. We then have a root datum of type $(I, \cdot)$ as in [11, 2.2.1], with $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = A_{ij}$ the corresponding symmetrizable Cartan matrix. For each $i \in I$, we define d_i to be the integer $\frac{1}{2}d(\alpha_i, \alpha_i)$. Then $d_i A_{ij} = \frac{1}{2}d(\alpha_i, \alpha_i)\left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right) = d(\alpha_i, \alpha_j)$ for each $i, j \in I$, and is thus a symmetric matrix over $\mathbb{Z}$. We use the same numbering as [1, Planches 1 to IX].

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v, and $\mathcal{A} \subseteq \mathbb{Q}(v)$ the ring $\mathbb{Z}[v, v^{-1}]$. For $N, M \in \mathbb{N}$ and $i \in I$ we put $v_i = v^{d_i}$ and define the following (which all lie in $\mathcal{A}$):

$$[N]_i = \frac{v_i^N - v_i^{-N}}{v_i - v_i^{-1}}, \quad [N]_i^! = [N]_i [N-1]_i \cdots [1]_i, \quad \begin{bmatrix} M \\ N \end{bmatrix}_i = \frac{[M]_i^!}{[N]_i^! [M-N]_i^!}.$$

We define the quantized enveloping algebra U corresponding to the above data (as in [11, 3.1.1 & 33.1.5]) to be the $\mathbb{Q}(v)$-algebra U with generators $1, E_1, E_2, \ldots, E_n$, $F_1, F_2, \ldots, F_n$, and K_μ for $\mu \in Y$, subject to the relations: (for each $i, j \in I$ and $\mu, \mu' \in Y$)

$$K_0 = 1,$$

$$K_\mu K_{\mu'} = K_{\mu + \mu'},$$

$$K_\mu E_i = v^{\alpha_i(\mu)} E_i K_\mu,$$

$$K_\mu F_i = v^{-\alpha_i(\mu)} F_i K_\mu,$$

$$E_i F_i - F_i E_i = \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}},$$

$$E_i F_j - F_j E_i = 0, \quad i \neq j,$$

$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \begin{bmatrix} 1 - A_{ij} \\ p' \end{bmatrix}_i E_i^p E_j E_i^{p'} = 0, \quad i \neq j,$$