EXCEPTIONAL OBJECTS IN HEREDITARY CATEGORIES

CLAUS MICHAEL RINGEL

Fakultät für Mathematik, Universität Bielefeld,
POBox 100 131, D-33 501 Bielefeld, Germany

ABSTRACT. Let k be a field and $\mathcal{A}$ a finite dimensional k-category which is a hereditary length category. We are going to show that the support algebra of any object of $\mathcal{A}$ without self-extension is a finite dimensional k-algebra. An object in $\mathcal{A}$ is said to be exceptional provided it is indecomposable and has no self-extensions. For an algebraically closed field k, Schofield has exhibited an algorithm for obtaining all exceptional objects starting from the simple ones. We will present a proof which works for arbitrary fields k.

Let $\mathcal{A}$ be an abelian category. The category $\mathcal{A}$ is said to be *hereditary* provided Ext^2 vanishes everywhere. Also, we recall that $\mathcal{A}$ is said to be a *length category* provided every object in $\mathcal{A}$ has finite length.

Let k be a field. We say that $\mathcal{A}$ is a *k-category* provided k operates centrally on all Hom-sets and such that the composition of maps is bilinear. Such a k-category is said to be *finite dimensional* provided the vector spaces $\mathrm{Hom}(X, Y)$ are finite dimensional, for all objects X, Y in $\mathcal{A}$.

Exceptional objects have been studied in various contexts. The terminology 'exceptional' was first used by Rudakov and his school [Ru] when dealing with vector bundles. The relevance of exceptional objects in the representation theory of finite dimensional hereditary k-algebras is well accepted; these exceptional modules are just the indecomposable partial tilting modules, they have also been called stones by Kerner [K1] and Schur modules by Unger [U]. We may refer to a recent survey of Kerner [K2] dealing with objects in hereditary length categories, or at least with representations of wild quivers.

The aim of this report is to focus attention to some interesting developments in the representation theory of finite dimensional hereditary algebras. This theory has an apparent combinatorial flavour; one of the reasons is the role the exceptional modules play. The existence of non-trivial finite dimensional modules without self-extensions should be considered as a feature which is peculiar to non-commutative representation theory. As we want to show, the existence of such modules seems also to be a kind of finiteness condition. We will present a proof of a very useful theorem of Schofield [S2], for an arbitrary base field k.

1991 *Mathematics Subject Classification.* Primary 16E60, Secondary 16G20. 16P20.

This result describes certain types of filtrations of an exceptional module X using as factors exceptional modules again; it is a kind of Jordan-Hölder theorem, but the classical unicity assertion is replaced by the assertion that there are precisely $s(X)-1$ essentially different kinds of filtrations, where $s(X)$ is the number of isomorphism classes of composition factors of X. Our presentation should be considered as a variation of the considerations by Crawley-Boevey [CB] dealing with a braid group operation on exceptional sequences, see also [R]. Along the way, we will focus attention to the so-called Bongartz complement of a sincere exceptional module.

1. Subfactors of objects without self-extensions.

Let $\mathcal{A}$ be an abelian category and A an object in $\mathcal{A}$. Let $A'' \subseteq A' \subseteq A$ be a chain of subobjects. Then A'/A'' is said to be a *subfactor* of A. If $\mathcal{U}$ is a subcategory of $\mathcal{A}$, we denote by $\mathcal{I}(\mathcal{U})$ the class of all subfactors of objects in $\mathcal{U}$.

Recall that for any object A of $\mathcal{A}$, one denotes by $\operatorname{add} A$ the full subcategory given by all direct summands of finite direct sums of copies of A.

Lemma 1.1. *Let $\mathcal{A}$ be a hereditary abelian category, and $\mathcal{U}$ a subcategory which is closed under extensions. Then $\mathcal{I}(\mathcal{U})$ is closed under extensions.*

Proof. Let A, B be objects in $\mathcal{U}$. Let $A'' \subseteq A' \subseteq A$ and $B'' \subseteq B' \subseteq B$ be chains of subobjects. Thus, A'/A'' and B'/B'' are subfactors of objects in $\mathcal{U}$, and we consider an extension: assume that there is given an exact sequence

$$0 \to A'/A'' \to C \to B'/B'' \to 0.$$

We construct stepwise the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A'/A'' & \longrightarrow & C & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A/A'' & \longrightarrow & D & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & B/B'' & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0
\end{array}$$