Exceptional Modules are Tree Modules

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ABSTRACT. We consider representations of a quiver over an arbitrary field. Recall that an indecomposable representation M without self-extensions is said to be exceptional. We are going to show that exceptional representations can be exhibited using matrices involving as coefficients just 0 and 1. Actually, if d is the dimension of M, there exists such a matrix presentation with precisely d–1 non-zero coefficients: the corresponding "coefficient quiver" is a tree.

Let k be a field, let Q be a (finite) quiver and kQ the path algebra of Q over k. Recall that a representation M of Q over k is of the form $M = (M_x, M_\alpha)_{x,\alpha}$; here, for every vertex x of Q, there is given a finite-dimensional k-space M_x, for every arrow $\alpha\colon x \to y$, there is given a linear transformation $M_\alpha\colon M_x \to M_y$. A representation M of Q over k (or better, the corresponding direct sum $\bigoplus_x M_x$) is just an arbitrary (finite-dimensional) kQ-module.

Let d_x be the dimension of M_x, and $d = \sum d_x$; we call d the *dimension* of M. A *basis* $\mathcal{B}$ of M is by definition a subset of the disjoint union of the various k-spaces M_x such that for any vertex x the set $\mathcal{B}_x = \mathcal{B} \cap M_x$ is a basis of M_x. Let us assume that such a basis $\mathcal{B}$ of M is given. For any arrow $\alpha\colon x \to y$, we may write M_α as a $(d_y \times d_x)$-matrix $M_{\alpha,\mathcal{B}}$ whose rows are indexed by $\mathcal{B}_y$ and whose columns are indexed by $\mathcal{B}_x$. We denote by $M_{\alpha,\mathcal{B}}(b, b')$ the corresponding matrix coefficients, where $b \in \mathcal{B}_x$, $b' \in \mathcal{B}_y$, these matrix coefficients $M_{\alpha,\mathcal{B}}(b, b')$ are defined by $M_\alpha(b) = \sum_{b' \in \mathcal{B}} M_{\alpha,\mathcal{B}}(b, b')\, b'$. By definition, the *coefficient quiver* $\Gamma(M, \mathcal{B})$ of M with respect to $\mathcal{B}$ has the set $\mathcal{B}$ as set of vertices, and there is an arrow (α, b, b') provided $M_{\alpha,\mathcal{B}}(b, b') \neq 0$. Note that there is a long-standing tradition in matrix theory to focus attention to such coefficient quivers, see e.g. [BR]. In his lectures [CB2] at the Banach center in Warsaw in 1988, Crawley-Boevey has drawn attention to the use of coefficient quivers for dealing with different types of matrix problems, as they arise in representation theory, in particular for dealing with representations of quivers.

We will call an indecomposable representation of Q over k a *tree module* provided there exists a basis $\mathcal{B}$ of M such that the coefficient quiver $\Gamma(M, \mathcal{B})$ is a tree. Note that for a tree module M of dimension d, there is a basis $\mathcal{B}$ of M such that precisely $d-1$ matrix coefficients are non-zero, and one may assume that all these coeffients are equal to 1 (see section 1). Thus, any tree module can be exhibited by 0-1-matrices.

We recall that an indecomposable representation M of Q over k is said to be *exceptional* (or to be a "partial tilting module", or a "stone") provided $\operatorname{Ext}^1_{kQ}(M, M) = 0$. The aim of this note is to present the following result:

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Theorem. *Let k be a field, let Q be a quiver. Any exceptional representation of Q over k is a tree module.*

The proof will use Schofield's inductive construction of all exceptional modules and a detailed consideration of the case of a directed quiver with 2 vertices (recall that a quiver is said to be *directed*, provided there is no directed cyclic path).

The first considerations below should be well-known, but there does not seem to exist a reference: We will show that we may restrict to the case of a directed quiver, since the support of any exceptional module is directed (for an exceptional representation M, we may define its *support* as the full subquiver given by all vertices x with $M_x \neq 0$).

On the other hand, starting with a directed quiver, all the preprojective and all the preinjective indecomposable representations are exceptional, thus Theorem above shows that they are tree modules. In particular, for the representation finite quivers, all the indecomposable representations are tree modules. In this special case, the result is due to Crawley-Boevey [CB2]. When Gabriel [G] exhibited matrix presentations of these modules, he used 0-1-matrices with few non-zero entries, but in four instances, the number of non-zero entries still can be reduced. Corresponding matrix presentations can be found in section 6.

1. Tree modules

Let M be a representation of Q over k. Here are some obvious properties of the coefficient quiver.

(1) *If M is indecomposable and $\mathcal{B}$ is a basis of M, then $\Gamma(M,\mathcal{B})$ is connected. If M is decomposable, then there exists a basis $\mathcal{B}$ of M such that $\Gamma(M,\mathcal{B})$ is not connected.*

Proof: Clearly, any decomposition of $\Gamma(M,\mathcal{B})$ yields a direct decomposition of M. Conversely, if we deal with a direct decomposition $M = M' \oplus M''$, and if $\mathcal{B}'$ is a basis of M' and $\mathcal{B}''$ a basis of M'', then let $\mathcal{B}$ be the disjoint union of the various sets $\mathcal{B}'_x$ and $\mathcal{B}''_x$. Clearly, $\mathcal{B}$ is a basis of M and $\Gamma(M,\mathcal{B})$ is the disjoint union of the quivers $\Gamma(M',\mathcal{B}')$ and $\Gamma(M'',\mathcal{B}'')$. (Note that the zero module is said to be decomposable, and the empty quiver is said to be not connected; thus the last assertion is valid also in the trivial case.)

Remark 1: Note that, in general, $\Gamma(M,\mathcal{B})$ *may be connected whereas M is decomposable.* For example, consider the representation $M = (k^2, k, [1,1])$ of the quiver $\circ \to \circ$, and let $\mathcal{B}$ be given by the canonical bases of k^2 and k. Then $\Gamma(M,\mathcal{B})$ is connected, but M is decomposable.

Remark 2: In general, *the shape of the coefficient quiver $\Gamma(M,\mathcal{B})$ not only depends on the representation M, but also on the choice of $\mathcal{B}$, even if we choose*