R-equivalence and special unitary groups

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The notion of R-equivalence, introduced by Manin in [6], is an important birational invariant of an algebraic variety defined over an arbitrary field. In the case of an algebraic group the set of R-equivalence classes has a natural group structure and it was studied by Colliot-Thélène and Sansuc in [2]. In particular, they computed this group for algebraic tori.

The group of R-equivalence classes for adjoint classical semisimple groups was computed in [9]. This computation enables to construct certain examples of non-rational adjoint groups. For simply connected semisimple groups the description of the group of R- equivalence classes is known only for inner forms of type A_n (Voskresenskii) and in some easy cases when this group is trivial (types C_n and G_2).

In the present paper we compute the group of R-equivalence classes for special unitary groups, i.e. for simply connected outer forms of type A_n. This computation comes as a result of the study of certain functorial properties of the group of R-equivalence classes for algebraic groups. To formulate these properties it is convenient to introduce the following category $\mathcal{C} = \mathcal{C}(F)$ for an arbitrary infinite field F. The objects of $\mathcal{C}$ are algebraic group homomorphisms $\alpha : G \to T$, where T is an algebraic torus over F and G is an algebraic group defined over F, such that $G(F)$ is dense in G and $G(F)/R = 1$. A morphism $(G \xrightarrow{\alpha} T) \to (G' \xrightarrow{\alpha'} T')$ in $\mathcal{C}$ is a group homomorphism $\beta : T \to T'$, such that for any field extension E/F the image of the composition $\beta_E \circ \alpha_E$ belongs to the image of $\alpha'_E : G'(E) \to T'(E)$.

It turns out (see section 2) that there is a natural functor from the category $\mathcal{C}$ to the category of abelian groups, taking $\alpha : G \to T$ to the group of R-equivalence classes $H(F)/R$, where $H = \ker(\alpha)$. As an application we show that this construction allows to define the *norm homomorphism* for groups of R-equivalence classes for certain algebraic groups. For example, if q is a non-degenerate quadratic form defined over F and E/F is a finite separable field extension, then there is a well-defined norm homomorphism for groups of R-equivalence classes of spinor groups

$$N_{E/F} : \mathrm{Spin}(q \otimes_F E)/R \longrightarrow \mathrm{Spin}(q)/R.$$

*The authors gratefully acknowledge the support of the Alexander von Humboldt-Stiftung and SFB 343 "Diskrete Strukturen in der Mathematik" and the hospitality of the University of Bielefeld.

Another application deals with the special unitary groups. Let L/F be a quadratic separable field extension, B be a central simple algebra over L with an involution τ of the second kind trivial on F. We prove (Theorem 4.3) that the group of R-equivalence classes of the special unitary group $\mathbf{SU}(B,\tau)$ is isomorphic to Σ/Σ', where Σ is the subgroup in $B^\times$ of all elements b, such that $\mathrm{Nrd}(b) \in F$ and Σ' is the subgroup in $B^\times$, generated by τ-symmetric elements. In particular, $SU(B,\tau)/R = 1$ if $\mathrm{ind}(B)$ is square-free.

The group Σ/Σ' does not depend on the involution τ. We explain this phenomenon in the second part of the paper. It turns out that the stable birational type of $\mathbf{SU}(B,\tau)$ depends only on the Brauer class of B (Corollary 5.7). We also prove (Theorem 5.4) that $\mathbf{SU}(B,\tau)$ is birationally isomorphic to $\mathbf{SU}(D,\sigma) \times \mathbf{A}_F^r$ for some r, where D is the central skewfield over L , Brauer equivalent to B, and σ is an involution of the second kind on D.

In particular, the group $\mathbf{SU}(B,\tau)$ is rational if $\mathrm{ind}(B) \leq 3$ (Corollary 5.5).

We use the following notation. For a vector space V over a field F and a field extension E/F we denote by V_E the E-vector space $V \otimes_F E$. For an algebraic variety X over F and for any F-scheme U the set $\mathrm{Mor}(U,X)$ of U-points of X is denoted by $X(U)$. If $U = \mathrm{Spec}(R)$, we simply write $X(R)$ for $X(U)$. For a morphism of varieties $\alpha : X \to Y$, the map of sets of points $X(R) \to Y(R)$ is denoted by α_R. $\mathbf{A}_F^n$ is the affine space over F. $\mathcal{AB}$ is the category of abelian groups.

1 The group of R-equivalence classes

Let F be an infinite field. Denote by $\mathcal{O}$ the semilocal subring in the field of rational functions $F(t)$ consisting of all functions defined at points $t = 0$ and $t = 1$. Evaluation at these points gives two F-algebra homomorphisms $e_i : \mathcal{O} \to F$, $i = 0, 1$.

Let X be an algebraic variety defined over a field F. For any element $x(t) \in X(\mathcal{O})$ and $i = 0, 1$ denote by $x(i)$ the *value of* $x(t)$ *at* i, i.e. the image of $x(t)$ under the map

$$X(e_i) : X(\mathcal{O}) \to X(F).$$

Lemma 1.1 *Let $\alpha : X \to Y$ be a morphism of algebraic varieties over a field F. Then*

$$\alpha_{\mathcal{O}}\big(x(t)\big)(i) = \alpha_F\big(x(i)\big)$$

for any $x(t) \in X(\mathcal{O})$ and $i = 0, 1$.

Proof. Follows from the commutativity of the diagram

$$\begin{array}{ccc} X(\mathcal{O}) & \xrightarrow{\alpha_{\mathcal{O}}} & Y(\mathcal{O}) \\ \downarrow X(e_i) & & \downarrow Y(e_i) \\ X(F) & \xrightarrow{\alpha_F} & Y(F). \end{array} \qquad \square$$