A family of extra-special 2-groups

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Let Γ be a Bieberbach group. By definition it is the torsion - free group given by a short exact sequence

$$(*)\ 1 \to Z^n \to \Gamma \to G \to 1,$$

where G is a finite group (called the holonomy group of Γ) and Z^n is a free abelian group of rank n which is a maximal abelian subgroup of Γ. Let $T : G \to GL(n, Z)$ be the representation which the group Γ defines on Z^n by conjugations. The rank of Γ is n. It is well known (see [1]) that Γ is isomorphic to a uniform discrete subgroup of the group $E(n)$ of the rigid motions of the n - dimensional Euclidean space $\mathbf{R^n}$. The orbit space $\mathbf{R^n}/\Gamma = M$ is a closed flat manifold, for short a flat manifold, with fundamental group Γ. In this note we shall define the sequence of extra - special 2-groups which are the epimorphic image of some Bieberbach groups. For the definition and properties of extra-special groups see [2] page 179.

Let Γ_{2n} be a (2n+1)-dimensional generalised **Hantzsche - Wendt** Bieberbach group defined by the short exact sequence

$$1 \to Z^{2n+1} \to \Gamma_{2n} \to (Z_2)^{2n} \to 1,$$

where $n \geq 1$ and the action of $(Z_2)^{2n}$ on Z^{2n+1} is defined by the $((2n+1) \times (2n+1))$ matrices:

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$$A_i = \left[\begin{array}{cccccccccc} -1 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -1 & ... & 0 & 0 & 0 & ... & 0 & 0 \\ ... \\ 0 & 0 & ... & -1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 & -1 & ... & 0 & 0 \\ ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & -1 & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & -1 \end{array} \right], 1 \leq i \leq 2n,$$

the "1" being in the i-th row and i-th column.
Moreover, the cocycle $s^\star \in H^2((Z_2)^{2n}, Z^{2n+1}) = H^1((Z_2)^{2n}, \mathbf{R^{2n+1}}/Z^{2n+1})$ correspondin the above exact sequence is given by the function $s : (Z_2)^{2n} \to \mathbf{R^{2n+1}}/Z^{2n+1}$; and

$$s(A_i) = u_i/2 + u_{i+1}/2 \text{ for } 1 \geq i \geq 2n$$

where $u_1, u_2, ..., u_n$ denotes a canonical base of Z^n. In [4] it was proven that Γ_{2n} is a Bieberbach group.

Let $T_{2n} \subset Z^{2n+1}$ be a subgroup generated by the following elements:

$$2u_1, (u_1 + (-u_2)), (u_2 + (-u_3)), ..., (u_{2n} + (-u_{2n+1})).$$

It is easy to see that T_{2n} is a subgroup of Z^{2n+1} of index 2. We shall prove that 2-groups $\mathcal{G}_{2n} = \Gamma_{2n}/T_{2n}, n \geq 1$, are extra-special 2-groups with known number of the involutions. From definition we have the following short exact sequence of finite groups:

$$0 \to Z_2 \to \mathcal{G}_{2n} \to (Z_2)^{2n} \to 0.$$

Hence we have $\mid \mathcal{G}_{2n} \mid = 2^{2n+1}$. For $n = 1$ $\mathcal{G}_2$ is the quaternion group of order 8, (see [3] the proof of the Proposition 3.5.). Let $\tau_{2n} = \sqrt{2}\cos\frac{\pi}{4}(2n+1)$. We have:

Theorem 1 *For each n, $\mathcal{G}_{2n}$ is an extra-special 2-group with exactly $2^{2n} + \tau_{2n}2^n - 1$ involutions.*

Proof: We shall consider Γ_{2n} as a subgroup of the isometry group $E(2n+1)$ of (2n+1)-dimensional euclidean space $\mathbf{R^{2n+1}}$, (see [4]). First we observe