DISCRETE ISOPERIMETRIC AND POINCARÉ-TYPE INEQUALITIES *

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Abstract

We study some discrete isoperimetric and Poincaré-type inequalities for product probability measures μ^n on the discrete cube $\{0,1\}^n$ and on the lattice $\mathbf{Z}^n$. In particular we prove sharp lower estimates for the product measures of 'boundaries' of arbitrary sets in the discrete cube. More generally, we characterize those probability distributions μ on $\mathbf{Z}$ which satisfy these inequalities on $\mathbf{Z}^n$. The class of these distributions can be described by a certain class of monotone transforms of the two-sided exponential measure. A similar characterization of distributions on $\mathbf{R}$ which satisfy Poincaré inequalities on the class of convex functions is proved in terms of variances of suprema of linear processes.

1 Introduction

For a number $p \in (0,1)$, let μ_p denote the Bernoulli measure on $\{0,1\}$ with probabilities $q = 1-p$ and p, respectively, and let μ_p^n denote the corresponding product measure on $\{0,1\}^n$. Denote by $s_i(x)$ the 'neighbour' of $x \in \{0,1\}^n$ obtained by changing the i-th coordinate: $[s_i(x)]_j = x_j$, for $j \neq i$, and $[s_i(x)]_i = 1 - x_i$. Furthermore, for any set $A \subset \{0,1\}^n$, define the function h_A on $\{0,1\}^n$ by

$$h_A(x) = \mathrm{card}\{i \leq n : x \in A,\ s_i(x) \notin A\}.$$

Define $C_p = \min(p,q)/\sqrt{pq}$ which ranges in the interval $[0,1]$ and $J(t) = t\sqrt{\log(1/t)}$, $t \geq 0$. The starting point of this paper are the following inequalities for the discrete cube due to M.Talagrand [T]. There exists an absolute constant $K > 0$ such that, for each subset A of $\{0,1\}^n$,

$$\int \sqrt{h_A(x)}\ d\mu_p^n(x)\ \geq\ \frac{C_p}{K}\, J\left(\mu_p^n(A)(1-\mu_p^n(A))\right), \tag{1.1}$$

$$\int \sqrt{h_A(x)}\ d\mu_p^n(x)\ \geq\ \sqrt{2}C_p\,\mu_p^n(A)\,(1-\mu_p^n(A)). \tag{1.2}$$

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Note that by the definition of $J(t)$ (1.2) immediately follows from (1.1) with a weaker bound involving the constant $1/(K\sqrt{\log 4})$.

For a comparison to the continuous case let γ_n denote the canonical Gaussian measure on $\mathbf{R}^n$, and let $\gamma_n^+(A)$ denote the Gaussian surface measure of a subset A of $\mathbf{R}^n$. According to a well-known inequality due to V.N.Sudakov, B.S.Tsirel'son and C.Borel ([ST], [Bor]), we have for any measurable set $A \subset \mathbf{R}^n$

$$\gamma_n^+(A) \geq I(\gamma_n(A)), \tag{1.3}$$

where the (so-called Gaussian isoperimetric) function I is given by $I(t) = \varphi(\Phi^{-1}(t))$, $t \in [0,1]$ (here Φ^{-1} is the inverse distribution function of $\Phi(x) = \gamma_1((-\infty, x])$ and $\varphi(x)$ denotes the density $\exp\{-x^2/2\}/\sqrt{2\pi}$). For the discrete cube, the quantity $\int \sqrt{h_A}\, d\mu_p^n$ measures the interior boundary $\partial_+ A = \{h_A > 0\}$ and thus it can be viewed as a discrete analogue of $\gamma_n^+(A)$. In addition, the expression $J(\mu_p^n(A)(1-\mu_p^n(A)))$ in (1.1) may be replaced by $I(\mu_p^n(A))$ since $c_1 I(t) \leq J(t(1-t)) \leq c_1 I(t)$, for some absolute constants $c_1, c_2 > 0$ and for all $t \in [0,1]$. Therefore, the inequality (1.1) itself may be viewed, up to a multiplicative absolute constant, as a discrete version of the Gaussian isoperimetric inequality (1.3).

Using (1.2), M.Talagrand obtained a quantitative version of Margulis' graph connectivity theorem [M]. G.A.Margulis proved that

$$\mu_p^n(\partial_+ A) \int h_A \, d\mu_p^n$$

is bounded from below by a quantity $c\,(p, \alpha)$ which depends on p and $\alpha = \mu_p^n(A)$ but not on n. It follows from (1.2) via a simple application of the Cauchy-Schwarz inequality to $\int \sqrt{h_A} = \int \sqrt{h_A}\, \mathbf{1}_{\partial_+ A}$ that $c(p,\alpha) = 2C_p^2(\alpha(1-\alpha))^2$. Note that this bound is small when pq is small since $pq \leq C_p^2 \leq 4pq$. We shall prove (c.f. Proposition 3.2) that (1.2) is valid in fact with a constant

$$C_p = \sqrt{\frac{2}{\max(p,q)} - 1}$$

which varies in the interval $[1, \sqrt{3}\,]$. Hence, the bound $c = c(p,\alpha)$ in Margulis' theorem can be chosen *independent* of p.

In Lemma 3.3 we shall provide a lower bound for the optimal constant C_p for inequalities of type (1.1) involving more general functions of $\mu_p^n(A)$ than $J(t(1-t))$, $t(1-t)$ and $I(t)$ discussed above and show e.g. that when p is close to 0 or 1 the optimal constant C_p in (1.1) is aproximately given by $1/\sqrt{\log \frac{1}{pq}}$.

Furthermore, we introduce the outer boundary $\partial_- A = \{h_{\bar{A}} > 0\}$, where $\bar{A} = \{0,1\}^n \setminus A$ denotes the complement of A in $\{0,1\}^n$, and associate to A another function

$$\bar{h}_A = h_A + h_{\bar{A}}.$$

The quantity $\int \sqrt{\bar{h}_A}\, d\mu_p^n$ will serve as a measure of the two-sided boundary $\partial A = \partial_+ A \cup \partial_- A$. Replacing the interior boundary function h_A by the two-sided boundary