On Perfect 3– and 4–Shift N–Designs

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Abstract. Necessary and sufficient conditions for the existence of perfect 3– and 4–shift N–designs are presented. From these perfect shift designs perfect run length limited codes capable of correcting single peak shifts of size 3 and 4 can be obtained. The results extend to the more general class of perfect $\mathcal{S}$– shift N–designs for sets $\mathcal{S} = \{1, a, \dots, a^r, b, \dots, b^s\}$.

Index terms. Peak-shift correction, perfect shift designs

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I. Introduction

In [1] Levenshtein and Vinck investigated perfect run–length–limited codes which are capable of correcting single peak shifts. As a basic combinatorial tool for the construction of such codes they introduced the concept of a *t–shift N–design*, which is defined to be an N–elementary subset $\mathcal{H} = \{h_1, \dots h_N\}$ of a finite additive Abelian group $\mathcal{G}$, with the property that for any $\sigma = 1, \dots t$ and any $i = 1, \dots, N$ all $2Nt$ elements $\pm\sigma \cdot h_i$ are different and not equal to zero.
Such a design is said to be *perfect* if $|\mathcal{G}| = 2tN + 1$, i. e., for every element $g \in \mathcal{G}$ there are exactly one $h_i \in \mathcal{H}$ and $\sigma \in \{1, \dots, t\}$ such that $g = \sigma \cdot h_i$ or $g = -\sigma \cdot h_i$.
Vinck and Levenshtein [1] also gave explicit constructions of perfect t–shift designs for the special values $t = 1, 2$ and for the case $t = \frac{p-1}{2}$, $N = (p^\nu - 1)/(p - 1)$, where p is a prime number.

In this note we shall investigate perfect t–shift N–designs in Z_{2tN+1}, the ring of integers modulo $2tN + 1$, for the parameters $t = 3$ and $t = 4$. In order to do so, we shall first introduce a more general shift design concept.
We call a subset $\mathcal{H} = \{h_1, \dots h_N\}$ of a finite (not necessarily additive) Abelian group $\mathcal{G}$ an *$\mathcal{S}$–shift N–design* if for any $\sigma \in \mathcal{S}$ and any $i = 1, \dots, N$ all $|\mathcal{S}| \cdot N$ elements $\sigma \cdot h_i$ are different and not equal to zero. Sometimes we shall omit the parameters $\mathcal{S}$ or N and only speak of a shift design or an $\mathcal{S}$–shift design.
So, the set $\mathcal{S}$ is no longer considered to be an interval. The t–shift designs in an additive Abelian group $\mathcal{G}'$ arise as a special case by considering $\mathcal{S}$–shift designs in the group $\mathcal{G} = \mathcal{G}' \backslash \{1, -1\}$, with the special choice $\mathcal{S} = \{1, \dots, t\}$ (since only one of the elements h and $-h$ can be contained in a design $\mathcal{H}$, we can identify them by considering the group $\mathcal{G}'$ modulo the units $\{1, -1\}$).
Perfectness is defined as before, i. e., every element $g \in \mathcal{G}$ has a unique representation $g = \sigma \cdot h$ for some $\sigma \in \mathcal{S}$ and $h \in \mathcal{H}$.
Observe that every perfect $\mathcal{S}$–shift N–design $\mathcal{H}$ in a group $\mathcal{G} = \mathcal{G}' \backslash \{1, -1\}$ yields a perfect $\mathcal{S}$–shift N–design $\mathcal{H}' = \mathcal{H} \cup (-\mathcal{H})$ in $\mathcal{G}'$.

In order to study 3– and 4– shift N–designs, we shall first investigate $\mathcal{S}$–shift N–designs for sets $\mathcal{S}$ of the form

$$\mathcal{S} = \{1, a, a^2, \dots, a^r, b, b^2, \dots b^s\}, \tag{1}$$

where $r > 1$ and $s > 1$ are nonnegative integers smaller than the orders of a and b in $\mathcal{G}$, respectively, which we shall denote by $\mathrm{ord}(a)$ and $\mathrm{ord}(b)$.
Observe that with $a = 2$ and $b = 3$ we obtain $\mathcal{S} = \{1, 2, 3\}$ for $r = s = 1$ and $\mathcal{S} = \{1, 2, 3, 4\}$ for $r = 2, s = 1$. Hence the 3–shift and 4–shift N–designs we want to study in this paper occur for a special choice of the parameters.
In order to characterize the structure of a perfect $\mathcal{S}$–shift N–designs with $\mathcal{S}$ defined as in (1), we first need some further notation. As usual, we denote by

$$< a, b >= \{a^i \cdot b^j, i = 1, \dots \mathrm{ord}(a), j = 1, \dots \mathrm{ord}(b)\} \tag{2}$$

the subgroup of $\mathcal{G}$ generated by the elements a and b. Furthermore let

$$\mathcal{F} = \{a^i \cdot b^j, i - j \equiv 0 \bmod (r + s + 1)\}. \tag{3}$$

Observe that $\mathcal{F}$ is the subgroup in $\mathcal{G}$ generated by the elements a^{r+s+1}, b^{r+s+1} and ab. It can be shown that a perfect $\mathcal{S}$–shift N–design in the group $\mathcal{G}$ exists exactly if for each element $f \in \mathcal{F}$ all possible representations $f = a^i \cdot b^j$ as a product of powers of a and b are such that $i - j \equiv 0 \bmod (r + s + 1)$. Necessary and sufficient conditions are given in the following theorem.

Theorem 1: Let $\mathcal{S} = \{1, a, a^2, \dots, a^r, b, b^2, \dots b^s\}$ be defined as in (1). A perfect $\mathcal{S}$–shift N–design in $\mathcal{G}$ is of the form

$$\mathcal{H} = \mathcal{F} \cup x_1 \cdot \mathcal{F} \cup \dots \cup x_{\rho-1} \cdot \mathcal{F}, \tag{4}$$