Asymptotic Properties of Additive Functionals of Brownian Motion

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Summary

In this paper, we study the asymptotic behavior of additive functionals of Brownian motion which are not necessarily of bounded variation. The result is then applied to the Hilbert transform of the Brownian local time.

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1 Introduction and Framework

A Borel measure μ on R^d is said to be in the Kato class K_d if

$$\text{(i)}\ \lim_{\alpha\to 0}\left[\sup_x \int_{|y-x|\le\alpha} \frac{|\mu|(dy)}{|x-y|^{d-2}}\right] = 0 \text{ when } d \ge 3.$$

$$\text{(ii)}\ \lim_{\alpha\to 0}\left[\sup_x \int_{|y-x|\le\alpha} |\mu|(dy)|\ln(|y-x|^{-1})\right] = 0, \text{ when } d = 2.$$

$$\text{(iii)}\ \sup_x \int_{|y-x|\le 1} |\mu|(dy) < +\infty \quad , \text{ when } d = 1.$$

Let $\rho \in H_2^1(R^d)$ with $|\nabla\rho|^2 \in K_d$, and let μ be a measure in the Kato class K_d. We introduce the following quadratic form on $L^2(R^d)$:

$$\left\{\begin{array}{lcl} Q(u,v) & = & \frac{1}{2}\int_{R^d} \nabla u(x)\cdot\nabla v(x)dx + \frac{1}{2}\int_{R^d}\nabla(uv)\nabla\rho(x)dx \\ & + & \int_{R^d} u(x)v(x)\mu(dx) \\ D(Q) & = & H_2^1(R^d) \quad \text{(the Sobolev space of order 1)} \end{array}\right. \tag{1.1}$$

This quadratic form is studied in [9] when $d \ge 3$. It is shown that $(Q, D(Q))$ is a closed semibounded form on $L^2(R^d)$. Although the results in [9] are stated for $d \ge 3$,they still hold for $d \le 2$. For our applications we would like to mention several facts in the case $d = 1$. It is well-known that for any $\epsilon > 0$, there exists $c(\epsilon)$ such that

$$|u(x)|^2 \le \epsilon \int_x^{x+1} u'(y)^2 dy + c(\epsilon)\int_x^{x+1} |u(y)|^2 dy\ ,\ u \in H_2^1(R)\ . \tag{1.2}$$

Given any $\mu \in K_d$; integrating both sides of (1.2) with respect to μ, we get

$$\int_R |u(x)|^2|\mu|(dx)$$

$$\le \epsilon\int_R u'(y)^2dy\int_{y-1}^{y}|\mu|(dx) + c(\epsilon)\int_R u^2(y)dy\int_{y-1}^{y}|\mu|(dx)$$

$$\le \epsilon\sup_x\int_{|x-y|\le 1}|\mu|(dy)\int_R u'(y)^2dy + c(\epsilon)\sup_x\int_{|x-y|\le 1}\mu(dy)\int_R u^2(y)dy \tag{1.3}$$