ESTIMATES FOR THE RAPID DECAY OF CONCENTRATION FUNCTIONS OF n-FOLD CONVOLUTIONS

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Abstract. We estimate the concentration functions of n–fold convolutions of one-dimensional probability measures. The Kolmogorov–Rogozin inequality implies that for non-degenerate distributions these functions decrease at least as $O(n^{-1/2})$. On the other hand, Esseen (1968) has shown that this rate is $o(n^{-1/2})$ iff the distribution has an infinite second moment. This statement was sharpened by Morozova (1977). Theorem 1 of this paper provides an improvement of Morozova's result. Moreover, we present more general estimates which imply the rates $o(n^{-1/2})$.

1. Introduction

Let us first introduce some notation. Let $\mathfrak{F}$ denote the set of probability distributions defined on the Borel σ-field of subsets of the real line $\mathbf{R}$, Φ_B the normal distribution with zero mean and variance B^2 $(B \geqslant 0)$, $\mathcal{L}(\xi) \in \mathfrak{F}$ the distribution of a random variable ξ, and $\mathbf{I}\{A\}$ the indicator function of an event A. For $F \in \mathfrak{F}$ the concentration function is defined by the equality $Q(F,\, b) = \sup_x F\big\{[x,\, x+b]\big\}$, $b \geqslant 0$. For $F, H \in \mathfrak{F}$ we denote the corresponding distribution functions by $F(x)$, $H(x)$; write $\rho(F,H) = \sup_x \big|F(x) - H(x)\big|$ for the Kolmogorov distance. Products and powers of measures will be understood in the convolution sense: $FH = F * H$, $H^m = H^{m*}$. For $F = \mathcal{L}(\xi) \in \mathfrak{F}$ we shall use the notation $\overline{F} = \mathcal{L}(-\xi)$ and $\widetilde{F} = F\,\overline{F}$. The distribution $\widetilde{F}$ is called symmetrized. Let $E_a \in \mathfrak{F}$ be the distribution concentrated at a point $a \in \mathbf{R}$, $E = E_0$. By $c(\,\cdot\,)$, $c_1(\,\cdot\,)$, $c_2(\,\cdot\,), \dots$ we shall denote positive quantities depending on the arguments only. The symbols $c, c_1, c_2, \dots$ will be used for absolute

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positive constants. Note that $c(\,\cdot\,)$, c can be different in different (or even in the same) formulas. We shall write $f \asymp g$ if $f \leqslant cg$ and $g \leqslant cf$. We denote by $\lceil \cdot \rceil$ the integer part of a number. The ends of proofs will be marked by $\square$.

This paper deals with the classical question about the rate of decay of $Q(F^n, b)$, $F \in \mathfrak{F}$, $b \geqslant 0$, as $n \to \infty$. Most of the known general bounds for $Q(F^n, b)$ decreases as $O(n^{-1/2})$. The Kolmogorov–Rogozin inequality (Rogozin (1961)) states that

$$Q(F^n, b) \leqslant \frac{c}{\sqrt{n\,(1 - Q(F, b))}}. \tag{1.1}$$

Esseen (1968) (see Petrov (1976), inequality (2.7) of Chapter III) strengthened (1.1) to

$$Q(F^n, b) \leqslant \frac{c}{\sqrt{n\,D(\widetilde{F}, b)}}, \tag{1.2}$$

where

$$D(F, b) = \int_{-\infty}^{\infty} \min\bigl\{x^2 b^{-2}, 1\bigr\}\, F\{dx\}, \qquad F \in \mathfrak{F}, \quad b > 0. \tag{1.3}$$

We can rewrite $D(F, b)$ in the form

$$D(F, b) = K(F, b) + G(F, b), \tag{1.4}$$

where

$$K(F, b) = b^{-2} \int\limits_{|x| \leqslant b} x^2\, F\{dx\} \qquad \text{and} \qquad G(F, b) = F\bigl\{\{x : |x| > b\}\bigr\}. \tag{1.5}$$

Esseen (1968) showed that $Q(F^n, b) = o(n^{-1/2})$ as $n \to \infty$ iff the distribution F has an infinite second moment. Concrete bounds of order $o(n^{-1/2})$ can be found in Esseen (1968), Mukhin (1972, 1973, 1976a, 1976b, 1989), Morozova (1977), Hall (1983), Griffin, Jain and Pruitt (1984), Suchkov and Ushakov (1989). However, most of these bounds have been proved assuming additional regularity-type conditions (see, e.g., (1.6)).

Arak (1981), Lemma 3.2, has found a powerful generalization of the Kolmogorov–Rogozin inequality (1.1) (see as well Arak and Zaitsev (1988), Theorem 4.2 of Chapter II). The right-hand side of Arak's inequality is expressed in terms of the distance of the support of F to a neighbourhood of the projection of a finite piece of a multidimensional lattice on the real line. This inequality may provide estimates of order $o(n^{-1/2})$.

The exact rate of decay of $Q(F^n, b)$ under the additional condition

$$\limsup_{y \to \infty} \frac{G(F, y)}{K(F, y)} < \infty \tag{1.6}$$