Exact value of module of trigonometric sums over a finite field for the polynomials of special type

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We described all the polynomials of special type which have the maximal value of the module of the trigonometric sum over finite fields.

Let $q = p^l,\ Q = q^m = p^{lm}$, where p is prime, l, m are positive integers, $m > 1,\ F_p \subseteq F_q \subset F_Q$ be the corresponding finite fields. For

$$f(x) = a_1 x + \cdots + a_n x^n,\ a_i \in F_Q,$$

define

$$S(f) = \sum_{x \in F_Q} exp\{\frac{2\pi i Tr_Q(f(x))}{p}\}, \tag{1}$$

where

$$Tr_Q(x) = x + x^p + x^{p^2} + \cdots + x^{p^{lm-1}},\ x \in F_Q.$$

We need the following presentation of $Tr_Q(x)$:

$$Tr_Q(x) = Tr_q(Tr_{Q\to q}(x)), \tag{2}$$

where

$$Tr_{Q\to q}(x) = x + x^q + \cdots + x^{q^{m-1}},\ x \in F_Q.$$

In this paper we study the values of the module of trigonometric sums for the polynomials of the following special type

$$f(x) = \sum_{s=0}^{\lfloor m/2 \rfloor} a_s x^{1+q^s}, \;\; a_s \in F_Q \;, \;\; Tr_Q(f(x)) \neq const \qquad (3)$$

(i.e. we exclude the trivial case when $|S(f)| = Q$). Of course, we might investigate polynomials from larger class

$$f'(x) = \sum_{i,j=0}^{m-1} a_{ij} x^{q^i + q^j},$$

but from the relation $Tr_{Q\to q}(y) = Tr_{Q\to q}(y^q), \;\; y \in F_Q$, it follows, that for any polynomial $f'(x)$ there is a polynomial $f(x)$ of the type (3), such that

$$Tr_{Q\to q}(f'(x)) \;=\; Tr_{Q\to q}(f(x))$$

and, therefore, $S(f') = S(f)$.

In general the investigation of the trigonometric sums of the polynomials of one variable over the field F_{q^m} reduces to the investigation of the multiple trigonometric sums of the polynomial of m variables over the field F_q. Indeed, let $\Phi(x)$ be an arbitrary polynomial over the field $F_Q, \; Q = q^m$. Fix a basis $\gamma_0, \gamma_1, \ldots, \gamma_{m-1}$ of F_Q over the field F_q , any element $x \in F_Q$ present in the form

$$x = \sum_{s=0}^{m-1} x_s \gamma_s \;\;, \;\; x_s \in F_q.$$

Define

$$G(x_0, x_1, \ldots, x_{m-1}\;) = \; Tr_{Q\to q}(\Phi(\sum_{s=0}^{m-1} x_s \gamma_s)) \;. \qquad (4)$$

It is clear, that $G(x_0, \ldots, x_{m-1})$ is a polynomial of m variables over the field F_q. By the definition of the multiple trigonometric sum

$$S_m(G(x_0, \ldots, x_{m-1})) = \sum_{x_0,\ldots,\; x_{m-1} \in \; F_q} exp\{\frac{2\pi i Tr_q(G(x_0, \ldots, x_{m-1}))}{p}\}$$

and using (4), we obtain that

$$S(\Phi(x)) = S_m(G(x_0, \ldots, x_{m-1})) \;.$$