Admissible vector fields and related diffusions on infinite dimensional manifolds

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§1 Introduction

The late 1960's saw the remarkable paper of Dalecky and Shnaiderman [DS69] constructing measures on certain infinite dimensional Lie groups via stochastic differential equations (with infinite dimensional noise) and discussing their quasi-invariance properties. Analysis on infinite dimensional spaces was also developing with discussions of differentiability of measures(cf. e.g. [DF91], [SW93]), a topic of increasing importance with continuing work by Professor Dalecky and his co-workers. In both of those advances, the idea of "admissible" derivatives plays an important role: in the curved space case the "derivatives" are given by vector fields. Such ideas are now fundamental to infinite dimensional analysis having been joined with developments in the work on H-differentiability by Gross[Gr67] and on Malliavin calculus[M76], with Driver's work[Dr92] leading the way to recent advances in analysis on path and loop space.

There have been many variations on the notion of "admissibility". Here we will describe that given in [EM96], showing how it leads to the construction of associated diffusions and Markov semi-groups on the state space. We will also show often infinite dimensional stochastic differential equations appear naturally: however in practice the coefficients are not regular enough for existence theorems in their present stage of development, so we have to rely on Dirichlet form theory(cf. [MOT94], [MR92]) to construct our processes. We give a class of concrete examples using results from [ELL 96], [ELL 97] to extend earlier work such as [DR 92].

§2 Admissible vector fields on $\mathcal{C}$-F manifolds

In this section we assume that M is a separable C^1 manifold modeled on a Banach space and equipped with a given Finsler structure τ ([AL78],[Pa66]). Let $TM := \cup_{x\in M} T_x M$ denote the tangent bundle of M.

We write $|v|_x = \tau(v)$ for $v \in T_x M$. Then $T_x M$ equipped with the norm $|\cdot|_x$ is a Banach space. Let f be a $\mathcal{C}^1$ map from M to another Finsler manifold N. We set

$$||df|| := \sup_{x\in M} ||df(x)||_{L(T_x M, T_{f(x)} N)}. \tag{2.1}$$

We write $f \in C_b^1(M;N)$ if $||df|| < \infty$. In particular we write $f \in C_1^1(M;N)$ if $||df|| < 1$. We introduce a pseudo metric ρ_M (Carathéodory metric[Ca69]) on M as follows

$$\rho_M(x,y) := \sup\{f(y) - f(x) : \ f \in C_1^1(M;R)\}. \tag{2.2}$$

Definition 2.1 We say that M is a **Caratheodory-Finsler manifold** (C-F manifold in short) (w.r.t. τ) if ρ_M is an admissible metric on M (i.e., ρ_M generates the original topology on M) and is complete.

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Below we collect some facts and examples concerning C-F manifolds without proof. For details see [EM96].

Proposition 2.2 ([EM96, 2.2]) Suppose that there exists a C-F manifold N and a closed embedding map $J \in C_b^1(M; N)$, then M is a C-F manifold.

The following results was communicated to us by C. J. Atkin.

Proposition 2.3 (Atkin) Suppose that the separable manifold M is modeled on a Banach space with a separable dual, then $\rho_m = d_M$ where d_M is the metric induced by the given Finsler structure. Thus if in addition d_M is complete, then M is a C-F manifold.

Example 2.4 (i) Let M be a separable Banach space with Finsler structure given by the Banach norm. then M is a C-F manifold.

(ii) Let M be a finite dimensional complete Riemannian manifold and the Finsler structure be given by the Riemannian metric. Then by Nash's embedding theorem M has a closed isometric embedding into a Euclidean space $\mathbb{R}^N$. Hence by (i) and proposition 2.2 M is a C-F manifold. See also Remark 2.5 below in this connection.

(iii) It follows directly from proposition 2.2 that any closed submanifold of a C-F manifold is again a C-F manifold.

The manifolds we are interested in will be C-F manifolds by the above results. They will be spaces of maps of a compact metric space S into a C-F manifold M. For example path or loop spaces if $S = [0, 1]$ or S^1. Set $E = C(S; M)$ the space of all continuous mappings with the compact open topology. Then E is a C^1 manifold (e.g. see [Pa 66], [Eli 66] for the case of M a Hilbert manifold and [Pe 67], [Ki 72] for more general M). Moreover, spaces of based loops and based paths are closed submanifolds of $C([0, 1]; M)$. For $\sigma \in E$, the tangent vector space $T_\sigma E$ can be identified with the space of all continuous map $v: S \to TM$ such that $v(s) \in T_{\sigma(s)}M$ for all $s \in S$. A natural Finsler structure on E is given by

$$|v|_\sigma := \sup_{\sigma \in S} |v(s)|_{\sigma(s)}, \ \forall \sigma \in E, \ v \in T_\sigma E. \tag{2.3}$$

One can easily check that $T_\sigma E$ equipped with the norm $|\cdot|_\sigma$ is a Banach space.

Proposition 2.5 (EM 96, 2.6) The mapping space E constructed above is a C-F manifold with respect to the Finsler structure given by (2.3), as are all its closed submanifolds with their induced Finsler structure.

In what follows let $E := C(S; M)$ be the mapping space specified in Proposition 2.5 or a closed submanifold of it. Let $\sigma \in E$ and $v \in T_\sigma E$. For $f \in C^1(E; R)$ we shall write $\partial_v f(\sigma) := df(v)(\sigma)$, the Fréchet derivative of f at σ along the direction v.

For a vector field v on E we write $\partial_v f(\sigma)$ for $\partial_{v(\sigma)} f(\sigma)$. Let $\mathcal{F}C_b^1(E)$ be the space of cylindrical functions in $C_b^1(E; R)$; so they are compositions of evaluations at a finite number of points of S with a C_b^1 function on a product of copies of M with its product Finsler. A monotone class argument shows that $\mathcal{F}C_b^1$ is dense in $L^2(E; \mu)$ for any finite Borel measure μ on E.

We shall always use the Borel σ-algebra of our manifolds. A vector field will be called **measurable** if $\partial_v f$ is measurable for all C^1 functions $f: E \to \mathbb{R}$. From now on we assume a finite measure μ is given on E. As usual we use f to denote the μ-a.s. equivalence class of a measurable function f on E

We need to specify a class of test functions for our Dirichlet forms. The standard example will be $\mathcal{F}C_b^1(E)$ but other choices could sometimes be more useful (e.g. see [EM 96] and below).