# Vector fields on mapping spaces and related Dirichlet forms and diffusions

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## 1 Introduction

A. Let $\mu$ be a Radon measure on an infinite dimensional smooth manifold $E$. Associated to $\mu$ there are various additional structures on $E$. This is seen from the example of Gaussian spaces where $E$ is a separable Banach space inducing an abstract Wiener space structure on $E$, from the example of path and loop spaces on finite dimensional Riemannian manifolds with measures induced by Brownian motions and Brownian bridges which are usefully analyzed using special "tangent spaces" [18], from the notions of "differentiability" of measures leading to classes of "admissible" vector fields describing the directions in which $\mu$ can be differentiated [4], and from very general considerations [11]. Here we describe a class of vector fields determined by $\mu$ and the differential structure of $E$ which also have a claim to be called "admissible" but are defined in terms of Dirichlet form theory rather than differentiability of $\mu$. Finite or suitably bounded countable families of such vector fields are shown to give rise to quasi-regular Dirichlet forms on $E$ with their associated diffusion processes, Markovian semigroups, and infinitesimal generators.

The ideas are valid for general separable metrizable manifolds but an adequately rich class of differentiable "test" functions is needed. Such would be assured if $E$ were modeled on a space admitting smooth partitions of unity with bounded derivatives. However this is not so for spaces of continuous paths (such as classical Wiener space) and for such mapping spaces it is often convenient to use cylindrical functions. On the other hand we wish to include such cases as iterated path spaces (paths on path spaces) and other examples

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of spaces of maps into infinite dimensional manifolds. To do this we introduce in Section 2 the notion of a Caratheodory-Finsler (C-F) manifold: a class of Finsler manifolds possessing a rich enough family of "test" functions. Closed submanifolds of separable Banach spaces, with induced Finsler structure are C-F manifolds, as is the space of continuous maps of a compact metric space into a C-F manifold. In this way we are able to give a unified treatment which covers and extends the existing results on path and loop spaces. We are also able to give a more detailed analysis of the structures involved, e.g. see the final parts of sections 3 and 4.

We go on to summarize our results in more detail. In particular subsection C,D which follow are summaries of what will be proved in sections 4 and 3 respectively.

B. Suppose $\mu$ is a finite measure and let $\mathcal{D}$ be a dense linear subspace of $L^2(E,\mu)$ consisting of $C^1$ maps $f : M \to R$. A Borel measurable vector field $v$ on $E$ is **$\mathcal{D}$-admissible** if

(i) $f \in \mathcal{D}$, $f = 0$ $\mu$-a.e. implies $\partial_v f = 0$ $\mu$-a.e.,

(ii) $\partial_v f \in L^2(E,\mu)$ for all $f \in \mathcal{D}$,

and

(iii) $\partial_v$, with domain $\mathcal{D}$, is a closable operator in $L^2(E,\mu)$.

Here $\partial_v f : E \to R$ is the Frechet derivative of $f$ in the direction $v$, i.e., $\partial_v f(\sigma) = df(v(\sigma))$, $\sigma \in E$.

If $v$ satisfies (i) and (ii) and possesses a divergence, i.e., there exists $\operatorname{div} v$ in $L^2(E,\mu)$ such that

$$\int_E \partial_v f(\sigma)\mu(d\sigma) = -\int_E f(\sigma) \operatorname{div} v(\sigma)\mu(d\sigma)$$

for all $f \in \mathcal{D}$, then $v$ will be said to be **strongly $\mathcal{D}$-admissible.** Strong admissibility implies admissibility if $\mathcal{D}$ is an algebra.

C. Now let $E$ be a closed submanifold (e.g. the manifold itself) of the manifold of continuous maps of a compact metric space $S$ into a C-F manifold $M$ and let $\mathcal{A}$ be a countable or finite family of $\mathcal{D}$-admissible vector fields on $E$ such that

$$\int_E \sum_{v\in\mathcal{A}} |\partial_v f(\sigma)|^2 \mu(d\sigma) < \infty$$

for all $f \in \mathcal{D}$. Then the form $(\mathcal{E},\mathcal{D})$ given by

$$\mathcal{E}(f,g) = \int_E \sum_{v\in\mathcal{A}} \partial_v f(\sigma)\partial_v g(\sigma)\mu(d\sigma)$$