Orthogonal Bases which Lead to Symmetric Nonnegative Matrices

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Abstract

In a paper dating back to 1983, Soules constructs from a positive vector x an orthogonal matrix R which has the property that for any nonnegative diagonal matrix Λ, with nonincreasing diagonal entries, the matrix $R\Lambda R^T$ has all its entries nonnegative. Independently, Fiedler in 1988 showed that any symmetric irreducible nonsingular matrix, whose powers are all M–matrices (and hence an MMA–matrix in the language of Friedland, Hershkowitz, and Schneider) must have an orthogonal matrix of eigenvectors $\tilde{R}$ which has similar properties to those of R.

Here, for a given positive n–vector x, we investigate the structure of **<u>all</u>** orthogonal matrices R for which, for any nonnegative diagonal matrix Λ as above, the matrices $R\Lambda R^T$ are nonnegative. Up to a permutation of its columns, each such R corresponds to a binary tree whose vertices are subsets of the set $\{1, 2, \ldots, n\}$ with the property that each vertex has either no successor or exactly two disjoint successors.

For such orthogonal matrices R and such nonsingular diagonal matrices Λ, we show that the set of matrices of the form $R\Lambda R^T$ and the set of inverse MMA–matrices (i.e. matrices whose inverses are MMA–matrices) coincide. Using this result we establish a relation between strictly ultrametric matrices and inverse MMA–matrices. Finally, we show that the QR–factorization of $R\Lambda R^T$, for certain such R's, has a special sign pattern.