# ON SMOOTH BEHAVIOUR OF PROBABILITY DISTRIBUTIONS UNDER POLYNOMIAL MAPPINGS

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Abstract. Let $X$ be a random variable with probability distribution $P_X$ concentrated on $[-1, 1]$ and $Q(x)$ be a polynomial of degree $k \geq 2$. The characteristic function of a random variable $Y = Q(X)$ is of order $O(1/|t|^{1/k})$ as $|t| \to \infty$ if $P_X$ is sufficiently smooth. In comparison for every $\varepsilon : 1/k > \varepsilon > 0$ there exists a singular distribution $P_X$ such that every convolution $P_X^{n\star}$ is also singular while the characteristic function of $Y$ is of order $O(1/|t|^{1/k-\varepsilon})$. While the characteristic function of $X$ is small when "averaged" the characteristic function of the polynomial transformation $Y$ of $X$ is *uniformly* small.

## 1. Introduction

Let $Q(x)$ denote a polynomial of degree $k \geq 2$:

$$Q(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_1 x. \tag{1-1}$$

Let $X$ be a random variable with a probability distribution $P_X$ concentrated on $[-1, 1]$. The characteristic function $g_Y(t)$ of a random varable $Y = Q(X)$ , i.e. $g_Y(t) = E\exp\{itY\}$ is of order $O(1/|t|^{1/k})$ as $|t| \to \infty$ when $P_X$ is sufficiently smooth (see Remark 1.3 below ).

In the first part of the paper we extend this result (see Theorem 1.1 and its Corollaries) and obtain in particular that under some conditions on the characteristic function $g_X$ we have

$$|g_Y(t)| = O(1/|t|^{\frac{1}{k}-\varepsilon}) \ \text{ as } \ |t| \to \infty, \tag{1-2}$$

where $0 \leq \varepsilon < 1/k$.

The second part is devoted to an example of $X$ with a singular distribution such that any finite number of convolutions with itself remains singular (see Lemma 3.3). Furthermore, since (1-2) holds for characteristic function of $Q(X)$ (see Corollary 4.3), its behaviour is similar to the case where $X$ has a smooth distribution.

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In the following we shall introduce some notions concerning the smoothing of distributions. We introduce a smoothing random variable $V$ independent of $X$ with probability density

$$p_V(x) = \frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2, \quad -\infty < x < \infty,$$

and characteristic function

$$g_V(t) = \begin{cases} 1-|t|, & |t| \le 1, \\ 0, & |t| > 1. \end{cases}$$

Fix any $T \ge 1$ and put

$$W_T = X + VT^{-1}.$$

Then

$$g_{W_T}(t) = \begin{cases} g_X(t)(1-|t|/T), & |t| \le T, \\ 0, & |t| > T. \end{cases}$$

Since $W_T$ converges weakly to $X$ as $T \to \infty$ , we have for any real $t$

$$E\, e^{itQ(W_T)} \to E\, e^{itQ(X)} \quad \text{as} \quad T \to \infty. \tag{1-3}$$

Moreover for any positive number $\eta$ we get

$$P\{|X + VT^{-1}| > 1 + \eta\} \le P\{|V| > \eta T\} \to 0 \quad \text{as} \quad T \to \infty.$$

Hence

$$\int_{|x|>1+\eta} e^{itQ(x)} p_{W_T}(x)dx \to 0 \quad \text{as} \quad T \to \infty, \tag{1-4}$$

where $p_{W_T}(x)$ is a probability density of $W_T$.
Combining (1-3) and (1-4) we conclude that in order to prove $|g_Y(t)| \le A(t)$ it suffices to show

$$\lim_{\eta\to 0}\lim_{T\to\infty} |J_\eta(t,T)| \le A(t), \tag{1-5}$$

where

$$J_\eta(t,T) = \int_{|x|\le 1+\eta} e^{itQ(x)} p_{W_T}(x)dx.$$

## 2. General Results and their Proofs

Recall that $g_X(t) = E\,\exp\{itX\}$. Put for $T > 0$

$$\phi_X(T) = \int_{-T}^{T} |g_X(t)|\, dt.$$