Harmonic and holomorphic Prym differentials on a compact Riemann surface

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Abstract

In 1980, R. C. Gunning studied the holomorphic (multiplicative) Prym differentials on a compact Riemann surface of genus $g \geq 2$ and their periods for the so-called essential characters. These period classes are an important set of transcendental invariants attached to the surface. Gunning introduced the cohomology bundle consisting of such period classes and the Prym bundle consisting of the holomorphic Prym differentials, and gave their explicit description for the case $g = 2$. In 1990, the author gave the explicit description of the Gunning bundle for surfaces of genus $g \geq 2$.

In the present article we introduce and study also the harmonic Prym bundle consisting of the harmonic Prym differentials for the case when $g \geq 2$. It is shown that the harmonic Prym bundle is the orthogonal sum of the holomorphic vector bundles of holomorphic Prym differentials and antiholomorphic Prym differentials. Some properties for the period classes of such differentials are established.

Introduction

The harmonic and holomorphic (multiplicative) Prym differentials on compact Riemann surfaces and their periods were investigated by F. Prym, and G. Rost [1] at the end of the last century; and by R. Koenig [2], O. Haupt [3], and H. Petersson [4] at the first half of our century. At present they are considered by many authors from different points of view: in connection with the study of Poincaré's series by C. Earle [5], and I. Kra [6]; in connection with application to number theory by G. Kempf [7], J. Fay [8], and J. Jorgenson [9]; and in variation theory by E. Jablow [10]. However, all these authors studied holomorphic Prym differentials as a rule with respect to two special kinds of complex one-dimensional representations (characters) of the fundamental group of a compact Riemann surface: when either the characters take their values only on the unit circle, or when they are equal to unit on half the generators.

In 1980, R. C. Gunning [11] studied the holomorphic Prym differentials and their periods for the so-called essential characters. The period classes of the holomorphic

Prym differentials on compact Riemann surfaces of genus $g \geq 2$ are an important set of transcendental invariants attached to the surfaces. Gunning introduced the cohomology bundle consisting of such period classes and the Prym bundle consisting of the holomorphic Prym differentials, and gave their explicit description for the case $g = 2$. The explicit description of the Gunning bundle for surfaces of genus $g \geq 2$ has been given in [12].

In the present article we introduce and study also the harmonic Prym bundle consisting of the harmonic Prym differentials for the case when $g \geq 2$. It will be shown that the harmonic Prym bundle is the orthogonal sum of the holomorphic vector bundles of holomorphic Prym differentials and of antiholomorphic Prym differentials. Some properties of the period classes for such differentials will be established.

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1. Preliminaries

Let $\left[F, \{\alpha_k, \beta_k\}_{k=1}^{g}\right]$ be a marked compact Riemann surface of genus $g \geq 2$, and Γ be a marked Fuchsian group of the first kind acting on the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$, so that U is the universal covering for F, and Γ is the group of covering (deck) transformations for the natural projection $\pi : U \to U/\Gamma = F$. The choice of generators for the fundamental group

$$\pi_1(F; O) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g : \prod_{j=1}^{g} [\alpha_j, \beta_j] = 1 \rangle$$

specifies the choice of generators for the group

$$\Gamma = \langle A_1, \ldots, A_g, B_1, \ldots, B_g : \prod_{j=1}^{g} C_j = 1 \rangle;$$

here O is a fixed base point on F; and $C_j = [A_j, B_j] = A_j B_j A_j^{-1} B_j^{-1}, \ j = 1, \ldots, g$.

Denote by $Hom(\Gamma, \mathbf{C}^*)$ the group of all one-dimensional representations ρ of Γ in $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ with the natural product operation; i.e., if $\rho_1, \rho_2, \rho \in Hom(\Gamma, \mathbf{C}^*)$, then $\rho_1 \cdot \rho_2 \in Hom(\Gamma, \mathbf{C}^*)$, where $(\rho_1 \cdot \rho_2)(T) = \rho_1(T)\rho_2(T)$ for all $T \in \Gamma$; and also $\rho^{-1} \in Hom(\Gamma, \mathbf{C}^*)$, where $(\rho^{-1})(T) = \rho(T)^{-1}$ for all $T \in \Gamma$.

Note that $Hom(\Gamma, \mathbf{C}^*)$ is a $2g$-dimensional complex analytic Lie group. Indeed, the map

$$\rho \to (\rho(A_1), \ldots, \rho(A_g), \rho(B_1), \ldots, \rho(B_g)) \in [\mathbf{C}^*]^{2g}$$

defines a global complex analytic structure on $Hom(\Gamma, \mathbf{C}^*)$. Therefore, $Hom(\Gamma, \mathbf{C}^*)$ is biholomorphically isomorphic to the group $[\mathbf{C}^*]^{2g}$ with the coordinate-wise product operation.