SIMPLICIAL DEGREES OF FUNCTORS

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Dedicated to the memory of Professor Roin Nadiradze

1.INTRODUCTION

The main result of this paper is to show that if G_* is a simplicial group of finite length, then H_nG_* also has finite length. Here the length of a simplicial group means the length of the corresponding Moore normalization and H_nG_* is a simplicial abelian group given by $[k] \mapsto H_nG_k$. A similar fact is true if we replace G_* by a simplicial ring and we take the algebraic K-functors instead of group homology.

The origin of such results goes back to the classical paper of Dold and Puppe (see Hilfsatz 4.23 of [DP]), where the following was proved: let $T : \mathbf{A} \to \mathbf{B}$ be a functor between abelian categories of degree d, meaning that the $(d+1)$-st cross-effect functor in the sense of Eilenberg and Mac Lane [EM] vanishes. Then $l(TX_*) \leq dl(X_*)$ for any simplicial object X_* in $\mathbf{A}$. Here $l(X_*)$ denotes the length of X_*. Actually, this property characterizes the degree of functors in the abelian case (see Lemma 3.6).

One can modify the notion of the cross-effects in the nonabelian framework (see [BaP]) and define the notion of degree of functors. One can also use the above inequality to define the simplicial degree in the nonabelian set-up. However in this way we get generally different invariants and the aim of this paper is to establish the relationship between the different notions. We show that many classical functors arising in homological algebra and K-theory have finite simplicial degree. Our Conjecture 4.7 claims that this should be always the case.

In the second section we deal with simplicial groups of finite length, where we essentially use the results of Conduché [C]. Based on [BaP], in the next section, we define the cross-effects and the notion of degree of functors in the nonabelian framework. In the section 4, we introduce simplicial degrees and give some inequalities relating all kinds of degrees. In section 5 we show our main result, that simplicial degree of n-dimensional group homology is n. In the last section we consider the algebraic K-functors and André-Quillen homology.

The author is very grateful to Tamar Datuashvili for the idea of considering this problem and for very useful discussions with her. I am very grateful to Friedhelm Waldhausen for his hospitality at the University of Bielefeld, while this paper was being written. The author was partially supported by the grant INTAS-93-2618.

2. SIMPLICIAL GROUPS OF FINITE LENGTH

Here we give the definition and some elementary facts about simplicial groups of finite lengths. We use the standard notation and terminology on simplicial objects (see [M]). Let $\mathcal{S}$ be the category of simplicial groups. We recall that for a simplicial group G_*, the *Moore normalization* NG_* is a chain complex of (nonabelian) groups defined by

$$N_nG_* = \bigcap_{0\leq i<n} Ker\ \partial_i,$$

whose boundary operator is the restriction of ∂_n. According to Lemma 5 and Proposition 2 of section 3 of ChII of [QHA], the functor

$$N_n : \mathcal{S} \to Groups$$

is exact, for any $n \geq 0$.

2.1. Definition. *We say that length of G_* is n if $N_iG_* = 0$, for $i > n$ and $N_nG_* \neq 0$.*

Let $\mathcal{S}^{(k)}$ be the category of such simplicial groups whose length is less or equal to k. We denote the length of G_* by $l(G_*)$. Obviously $l(G'_*) \leq l(G_*)$ if $G'_* \subset G_*$. Moreover, for any short exact sequence of simplicial groups

$$0 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 0$$

one has

$$l(G_*) = max\{l(G^{'}_*), l(G^{''}_*)\}. \tag{2.1.1}$$

This is a consequence of the exactness of the functor N_n. We shall use (2.1.1) very extensively in the following situations. Assuming one has a spectral sequence in the category of simplicial objects: $E^r_{pq} \Longrightarrow A_{p+q}$. Then (2.1.1) has following consequences: a) if $l(E^r_{p,n-p}) \leq d$ for any p, then $l(A_n) \leq d$ and b) if $l(A_n) < \infty, n \geq 0$ and $l(E^r_{pq}) < \infty$ for $q \neq k$, then $l(E^r_{pk}) < \infty$ as well.

2.2. Examples and notations. i) For any group G, one denotes by $K(G,0)$ the constant simplicial group whose value is G. Then $l(K(G,0)) = 0$. For any abelian group A, one denotes by $K(A,n)$ the simplicial abelian group, whose normalization is A in the dimension n and 0 in other dimension. Thanks to the Dold-Puppe equivalence, such object is unique up to isomorphism. By definition $l(K(G,n)) = n$.