ASYMPTOTIC DISTRIBUTION OF QUADRATIC FORMS I

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Abstract. We consider quadratic forms

$$Q_n = \sum_{1 \le j < k \le n} a_{jk} X_j X_k,$$

where X_j are i.i.d. random variables with finite third moment. We obtain optimal bounds for the Kolmogorov distance between the distribution of Q_n and the distribution of the same quadratic forms with X_j replaced by corresponding Gaussian random variables.

1. Introduction and results

Let $X_1, X_2, \ldots$ denote independent random variables (r.v.) such that $\mathbf{E}\, X_j = 0$ and $\mathbf{E}\, X_j^2 = 1$, $j = 1, 2, \ldots$. Let $A = A^{(n)} = \{a_{njk}\}_{j,k=1}^{n}$ denote a symmetric matrix with eigenvalues $\lambda_{n1}, \ldots, \lambda_{nn}$ ordered to be non increasing in absolute value. Throughout we shall assume that

$$\sum_{j,k=1}^{n} a_{njk}^2 = \sum_{j=1}^{n} \lambda_{nj}^2 = 1. \tag{1.1}$$

Consider the quadratic forms (q.f.)

$$Q_n = \sum_{j,k=1}^{n} a_{njk} X_j X_k \qquad \text{and} \qquad G_n = \sum_{j,k=1}^{n} a_{njk} Y_j Y_k,$$

where $Y_1, Y_2, \ldots$ are orthonormal Gaussian r.v.'s.

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There exists an extensive literature on different probability problems for quadratic forms. One of the most important problems is the investigation of the asymptotic of distributions of quadratic forms G_n and Q_n. Sevastyanov (1961) described the class of distributions which are limits of G_n as $n \to \infty$. Whittle (1960, 1964) obtained inequalities for moments of Q_n similar to the Rosenthal inequalities for sums of independent r.v.'s and proved a Central Limit Theorem (CLT) under the additional condition that the matrices $A^{(n)}$ have a special quasi-diagonal structure. Varberg (1966) obtained conditions of convergence of Q_n in quadratic mean and almost surely and investigated weak convergence of G_n too. There is a number of papers on the convergence of quadratic forms Q_n and especially of G_n to various limit distributions. See, for example, de Jong (1987) (CLT for quadratic forms), Fox and Taqqu (1985, 1987) (limit theorems for quadratic forms G_n of Gaussian r.v.'s with long-range dependence), Mikosch (1990, 1991) (functional CLT for quadratic forms, law of iterated logarithm). De Wet and Venter (1973) gave conditions under which the distribution of a quadratic form Q_n converges to that of $\sum_m \alpha_m (Y_m^2 - 1)$, where $\{\alpha_m\}$ is a sequence of real numbers.

Rotar' (1973) proved that under sufficiently weak conditions on the matrix A and for large n the distribution of Q_n is close to the distribution of G_n. Rotar' and Gamkrelidze (1977) obtained bounds for the error of this approximation which were improved by Rotar' and Shervashidze (1985). Below we formulate their result. We shall use the following notation:

$$F_j(x) = \mathbf{P}\{X_j \le x\}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-u^2/2\}\, du,$$

$$\nu_j = 3 \int_{-\infty}^{\infty} x^2 \left| F_j(x) - \Phi(x) \right| dx, \qquad s_j^2 = \sum_{k=1}^{n} a_{njk}^2,$$

$$L = \sum_{j=1}^{n} \nu_j s_j^3 + \sum_{j,k=1}^{n} \nu_j \nu_k \big| a_{njk} \big|^3, \qquad \Delta = \sum_{j=1}^{n} \lambda_{nj}^4.$$

Here and in what follows we shall consider matrices A with zero diagonal only, that is, $a_{njj} = 0$ for $1 \le j \le n$.

Theorem (Rotar', Shervashidze (1985)). *Let* $\Delta < 1/2$. *Then*

$$\sup_x \Big| \mathbf{P}\big\{ Q_n \le x \big\} - \mathbf{P}\big\{ G_n \le x \big\} \Big| \le C \big(1 - \log(1 - 2\Delta) \big)^{3/4} L^{1/4},$$

where C *is an absolute constant.*

The aim of our paper is to improve this bound for independent identically distributed (i.i.d.) $X_1, X_2, \ldots, X_n$ under various assumptions on the spectra of the matrices A. We apply these results in particular to investigate the rate of convergence of the distributions of Q_n to Gaussian distributions.

Write

$$\delta_n(A, F) = \sup_x \Big| \mathbf{P}\big\{ Q_n \le x \big\} - \mathbf{P}\big\{ G_n \le x \big\} \Big|,$$