INFINITE PROJECTION PROPERTIES

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Abstract. A structure $\mathfrak{A}$ is κ-projective if the projections are the only retraction of $\mathfrak{A}^\kappa$ on its diagonal. We investigate projectivity with respect to infinite exponents.

1. Introduction

The definition of κ-projectivity (which makes sense in any category with products) extends a notion introduced by E. Corominas [Cor]. Given a structure $\mathfrak{A}$ and a cardinal κ, denoting by $\delta : \mathfrak{A} \to \mathfrak{A}^\kappa$ the diagonal embedding, say that $\mathfrak{A}$ is *κ-projective* if the projections are the only morphisms $f : \mathfrak{A}^\kappa \to \mathfrak{A}$ satisfying $f \circ \delta = \mathrm{id}_{\mathfrak{A}}$. Reflexive κ-projective structures enjoy the following irreducibility property (see [Cor]) : if such a structure is a retract of a product $\prod_{\alpha<\kappa} \mathfrak{A}_\alpha$ then it is a retract of at least one $\mathfrak{A}_\alpha$. (A structure $\mathfrak{A}$ on a set A is -weakly- reflexive if the constant selfmappings of A are all endomorphisms of $\mathfrak{A}$.)

A κ-projective structure is obviously κ'-projective for every $\kappa' < \kappa$. As for the opposite direction, it was shown in [P,R,S] and [D,McK,N,P] that, provided it is not "2-elementary-group-like", a reflexive 2-projective structure is *finitely-projective*, ie. it is n-projective for each integer n. The question of the existence of countable, ω-projective and reflexive multirelations was raised in [P,R,S], following the observation that a first order structure $\mathfrak{A}$ on a finite set A (with $|A| \geq 2$) can be κ-projective for no infinite κ (consider $\lim_{\mathcal{U}} : A^\kappa \to A$ for some non-principal ultrafilter $\mathcal{U}$ on κ) ; note that this argument also holds for any *topologically compact* structure $\mathfrak{A} = (A; \dots)$ (ie. such that there exists a Hausdorff-compact topology on A, for which the relations and functions of $\mathfrak{A}$ are respectively closed and countinuous with respect to the correponding product topologies). More generally, atomic compact structures can be κ-projective for no infinite κ either, as they are precisely the structures which are retracts of all their ultrapowers -with respect to the diagonal embedding- (a first order structure $\mathfrak{A}$ on a set A is atomic compact, if each of its atomic systems with parameters in A is solvable, provided it is finitely solvable). Conversely, given a finitely-projective structure $\mathfrak{A}$ and a cardinal κ, for each retraction f of $\mathfrak{A}^\kappa$ on its diagonal, there exists a ultrafilter $\mathcal{U}_f$ on κ, such that f and $\lim_{\mathcal{U}_f}$ coincide on the κ-tuples which take only finitely many distinct values [Weh]. Observe that such an f yields a retraction of the ultrapower $\mathfrak{A}^\kappa/\mathcal{U}_f$ on its diagonal precisely when it is quotientable by the ultrafilter $\mathcal{U}_f$, ie. when any two couples coinciding on an element of $\mathcal{U}_f$ get the same image by f ; in this case, either $\mathcal{U}_f$ is principal and f is a projection, or $\mathfrak{A}^\kappa/\mathcal{U}_f$ is ω^+-saturated and $\mathfrak{A}$ is compact with respect to countable atomic systems with parameters. But in general, f need not be quotientable by $\mathcal{U}_f$. However, for certain classes of structures, such retractions are always prequotientable by their associated ultrafilter, ie. f and $\lim_{\mathcal{U}_f}$ coincide on each κ-tuple which is constant on some element of $\mathcal{U}_f$. Under this assumption, we shall relate some deficits of atomic compactness to restrictions on the ultrafilters $\mathcal{U}_f$ (lemma 2), eventually getting general criteria of infinite projectivity, that we illustrate with examples mainly taken from binary relational structures.

2. Preliminaries

2.1. **Definitions and conventions.** Given a set A, a cardinal κ, a family $\mathcal{K} = (K_\xi : \xi < \lambda)$ of pairwise disjoint subsets of κ and $\mathbf{a} \in A^{\kappa \setminus \cup\{K_\xi : \xi < \lambda\}}$, for every λ-tuple $(a_\xi : \xi < \lambda) \in A^\lambda$, denote by $\mathbf{a} \otimes \bigotimes_{\xi<\lambda} a_\xi^{K_\xi}$ the the κ-tuple which takes the value a_ξ on each K_ξ and coincides with $\mathbf{a}$ elsewhere ; to each $f : A^\kappa \to A$, associate the selfmapping of A, $f^{\mathbf{a}}_{\mathcal{K}} : (a_\xi : \xi < \lambda) \mapsto f(\mathbf{a} \otimes \bigotimes_{\xi<\lambda} a_\xi^{K_\xi})$ (we shall freely elaborate these notations).

Given a first order structure $\mathfrak{A} = (A; \dots)$ over a first order language $\mathcal{L}$, let $\mathcal{L}_A$ denote the expansion of $\mathcal{L}$ by A as a set of new constant symbols. So, the *retractions* of $\mathfrak{A}^\kappa$ on its diagonal (which we shall also just call the *κ-retractions* of $\mathfrak{A}$) are precisely the $\mathcal{L}_A$-morphisms $f : \mathfrak{A}^\kappa \to \mathfrak{A}$, where each $a \in A$ is interpreted by a^κ in A^κ. Then $\mathfrak{A}$ is *κ-projective* if and only if every such retraction coincides with a *projection* $p_\alpha : (a_\beta : \beta < \kappa) \mapsto a_\alpha$.

Let us say that $\mathfrak{A}$ is *reflexive* if for each atomic $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ (with all free variables displayed) and every $a \in A$, $\mathfrak{A} \models \varphi(a, \dots, a)$. In particular, a relational structure $\mathfrak{A} = (A; \mathcal{R}_i : i \in I)$ is reflexive if and only if $(a, \dots, a) \in \mathcal{R}_i$ for each $i \in I$ and every $a \in A$ (each $\mathcal{R}_i \subset A^{n_i}$ interprets the corresponding n_i-ary relational symbol r_i).

We shall mainly be interested in κ-projectivity for reflexive structures (cf. the related notion of κ-strong rigidity [Rosb]). Some of them, such as the homogeneous structures, may have many endomorphisms (see 6.3).

2.2. **Examples.** Let us first mention that the conclusion of the theorem in [P,R,S] and [D,McK,N,P] applies to reflexive binary relations and metric spaces : 2-projective ones are indeed finitely-projective.

2.2.1. *Graphs.* Our *graphs* will be the binary relations $\mathfrak{G} = (A; \mathcal{R})$ which are reflexive and symmetric. One easily sees that a 2-projective graph must be connected and have no *dangling tree* (the dangling trees of $\mathfrak{G}$ are its induced subgraphs which contain no cycle and are linked to the rest of the graph by exactly one edge). Conversely, if such a graph contains no triangle, then it is 2-projective (cf. [Haz]).

2.2.2. *Metric spaces.* To each metric space $\mathfrak{X} = (A, d)$, one can associate a reflexive binary relational structure $\mathfrak{M}_{\mathfrak{X}} = (A; \mathcal{D}_r : r \in \mathbb{Q}_+)$ in such a way that the non-expansive mappings between metric spaces are exactly the homomorphisms between the corresponding relational structures : for each $r \in \mathbb{Q}_+$ and every a and b in A, $(a, b) \in \mathcal{D}_r$ if and only if $d(a, b) \leq r$ (note that the distance can be recovered ; indeed $d(a, b) = \sup\{r \in \mathbb{Q}_+ : (a, b) \in \mathcal{D}_r\}$.) Hence, a metric space is κ-projective (with respect to the non-expansive mappings) if and only if its associated relational structure is. In the same way, a connected graph $\mathfrak{G} = (A; \mathcal{R})$ is κ-projective if and only if the metric space $\mathfrak{X}_{\mathfrak{G}} = (A, d)$ is, with $d : A \times A \to \mathbb{N}$ denoting the "shortest path distance". Furthermore, $\mathfrak{X}_{\mathfrak{G}}$ embeds in the metric space obtained by "materializing" each edge of $\mathfrak{G}$ by a segment of length 1 ; it can be shown (cf. [Del2]) that this new metric space is κ-projective if and only if $\mathfrak{G}$ is. For example the circle $\mathfrak{C}$ of length 1 equipped with its geodetic distance (ie. $\mathbb{R}/\mathbb{Z}$) is