PROJECTION PROPERTIES AND BINARY RELATIONS

CHRISTIAN DELHOMMÉ

ABSTRACT. We establish general criteria of "projectivity" for reflexive binary relations. We obtain in particular, for each integer n, the existence of n-strongly projective graphs.

1. Introduction

Consider a structure $\mathfrak{M}$ for which constant mappings are endomorphisms. Given an integer n, the structure is *n-strongly projective* (for short, $\mathfrak{M}$ is n-SP) if projections are its only n-ary operations (regular with respect to each argument) which are identical on the diagonal ; it is *n-weakly projective* (for short, $\mathfrak{M}$ is n-WP) if projections are the only retractions of $\mathfrak{M}^n$ on its diagonal. Projective structures, and especially projective binary relational structures arise in diverse fields (eg. ordered sets [Cor], clone theory [P,R,S], social sciences [Pou], [P,R,S], algebraic compactness [Weh] ...) Weak projectivity has been studied in a universal algebra framework [P,R,S], [D,McK,N,P] : It was shown that 3-WP implies n-WP for all n's, and follows from 2-WP except for the 2-elementary groups. Strong projectivity does not behave in such a uniform way : for instance, a finite graph (reflexive and symmetric binary relation) can be n-SP only for finitely many n's [Del1].

In the following, we shall be concerned with binary relations, for which we establish criteria of projectivity, based on elementary techniques of algebraic topology. Here is how we proceed :

Given a integer n and a connected reflexive binary relation $\mathfrak{R}$ based on some set V, let $\delta : V \to V^n$ denote the diagonal injection and let, for each $\mathbf{a} \in V$ and $k \in \{1, \cdots, n\}$, $\iota_k^{\mathbf{a}} : V \to V^n$ denote the k^{th} marginal injection at $\mathbf{a}$. Consider $\mathfrak{R}^{(n)}$, the "smallest" of those relations $\mathfrak{P}$ based on V^n with the property that δ and the all the $\iota_k^{\mathbf{a}}$'s are morphisms from $\mathfrak{R}$ to $\mathfrak{P}$ (in particular, they induce homomorphisms δ_* and $\iota_{k*}^{\mathbf{a}} : Z\mathfrak{R} \to Z\mathfrak{R}^{(n)}$ between the corresponding cycle spaces). We define $L_{\mathfrak{R}}^{\square n}$ and $T_{\mathfrak{R}}^{(n)}$, two subsets of $Z\mathfrak{R}^{(n)}$ such that (cf. lemma 1), the morphisms $\tau_{L_{\mathfrak{R}}^{\square n}} \circ \iota_{k*}^{\mathbf{a}} : Z\mathfrak{R} \to Z\mathfrak{R}^{(n)} / \langle L_{\mathfrak{R}}^{\square n} \rangle$ depend only on k (with $\tau_{L_{\mathfrak{R}}^{\square n}}$ denoting the canonical surjection from $Z\mathfrak{R}^{(n)}$ on its quotient by its subgroup generated by $L_{\mathfrak{R}}^{\square n}$), and moreover, $\iota_{1\sharp} + \cdots + \iota_{n\sharp} = \tau_{L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)}} \circ \delta : Z\mathfrak{R} \to Z\mathfrak{R}^{(n)} / \langle L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)} \rangle$ (each $\iota_{k\sharp}$ denotes any $\tau_{L_{\mathfrak{R}}^{\square n} \cup T_{\mathfrak{R}}^{(n)}} \circ \iota_{k*}^{\mathbf{a}}$) : Indeed, let $\mathfrak{R}_*$ be the 1-dimentional complex associated to $\mathfrak{R}$, and let $\mathfrak{R}_*^{\otimes n}$ be its n^{th} tensorial power ; note that for each k, the $\iota_k^{\mathbf{a}}$'s define chain mappings from $\mathfrak{R}_*$ to $\mathfrak{R}_*^{\otimes n}$ which are homotopic with one another ; $L_{\mathfrak{R}}^{\square n}$ is then the set of boundaries of the 2-dimensional cells of $\mathfrak{R}_*^{\otimes n}$ (each such cell is the "product of 2 edges and $n-2$ vertices of $\mathfrak{R}$", and in particular, its length is equal to 4) ; as for $T_{\mathfrak{R}}^{(n)}$, it can be any subset of $Z\mathfrak{R}^{(n)}$ containing, for each diagonal edge, at least one cycle which is the "sum of this edge with a 1-chain of $\mathfrak{R}_*^{\otimes n}$" (the minimum length of such cycles is $n+1$). Now consider an n-ary operation F of $\mathfrak{R}$

such that $F \circ \delta = \mathrm{id}_V$, a relation $\mathfrak{P}$ on V^n containing all edges of $\mathfrak{R}^{(n)}$ (so that in particular, the identity mapping of V^n is a morphism from $\mathfrak{R}^{(n)}$ to $\mathfrak{P}$), a subset P of $Z\mathfrak{P}$ such that $L^{\square n}_{\mathfrak{R}} \cup T^{(n)}_{\mathfrak{R}} \subset \langle P \rangle$ and a subset R of $Z\mathfrak{R}$. Assume that F is a morphism from $\mathfrak{P}$ to $\mathfrak{R}$ and, denoting by $F_* : Z\mathfrak{P} \to Z\mathfrak{R}$ the induced morphism, that $F_*(P) \subset \langle R \rangle$. Then the morphisms $\tau_R \circ F_* \circ \iota^{\mathbf{a}}_{k*} : Z\mathfrak{R} \to Z\mathfrak{R}/\langle R \rangle$ depend only on k (in that case denote them by $F_{k\sharp}$) ; moreover, they satisfy : $F_{1\sharp} + \cdots + F_{n\sharp} = \tau_R$. It is then the object of proposition 2 to conclude that some $F_{k\sharp}$ must be equal to τ_R, provided that $\mathfrak{R}$ satisfies some condition of stability with respect to R (which roughly expresses that $\langle R \rangle$ is not "too big" compared to R —in particular, $\mathfrak{R}$ is always $\varnothing$-stable—) : the key steps of the proof consist, given a "minimal" cycle γ outside R, in checking that for some k_0, $F_{k_0\sharp} \cdot \gamma = \tau_R \cdot \gamma$, and then in observing that, for any $k \neq k_0$, $F_{k\sharp}(Z\mathfrak{R}) \subset \mathbb{Z} \cdot \tau_R \cdot \sigma$. In particular, if moreover the identity mapping of V is the only endomorphism f of $\mathfrak{R}$ for which $\tau_R \circ f_* = \tau_R$ (in this case say that $\mathfrak{R}$ is R-tight), then the operation F had to be a projection (cf. corollary 2).

Summing up, given $\mathfrak{R}$, n, F, $\mathfrak{P}$, P and R, if the following four condition are met, then F is a projection :

— F is a morphism from $\mathfrak{P}$ to $\mathfrak{R}$,
— $L^{\square n}_{\mathfrak{R}} \cup T^{(n)}_{\mathfrak{R}} \subset \langle P \rangle$,
— $F(P) \subset \langle R \rangle$,
— $\mathfrak{R}$ is R-stable and R-tight.

(Observe that, when $R = \varnothing$, the last condition is by no means restrictive for our purpose, as it is essentially required for $\mathfrak{R}$ to be projective -cf. fact 1 and lemma 3-, provided it is not a "cycle".)

Now given a "class $\mathcal{F}$ of operations", the sufficient conditons that we get for an $\mathfrak{R}$ to be "projective with respect ot this class" express that $\mathfrak{R}$ is R-stable and R-tight for some R's related to $\mathcal{F}$ via some couples $(\mathfrak{P}, P)$'s. Note that, starting with the R's and proceeding backwards, one may eventually enlarge the given class $\mathcal{F}$. Double-fine projectivity, for instance, can be introduced in such a way starting from weak projectivity.

The sets $L^{\square n}_{\mathfrak{R}}$ and $T^{(n)}_{\mathfrak{R}}$ are introduced in section 3. Proposition 1 in the folowing section is a weak version of proposition 2, which does not require the developpements about tightness (indeed rigidity) and stability. Theorem 1, which is obtained as an immediate corollary, yields in particular the existence of strongly 3-projective graphs.

2. Preliminaries

2.1. **Reflexive binary relations.** A *reflexive binary relation* on a set V is any couple $\mathfrak{R} = (V, \mathcal{E})$ such that $\Delta_V \subset \mathcal{E} \subset V \times V$, where Δ_V denotes the set $\{(x,x) : x \in V\}$; we shall write $x \sim_{\mathfrak{R}} y$ for $(x,y) \in \mathcal{E}$. By "relation", we shall always mean "reflexive binary relation". The relation $\mathfrak{R}$ is a *graph* when it is moreover *symmetric*, ie. when $x \sim_{\mathfrak{R}} y$ is equivalent to $y \sim_{\mathfrak{R}} x$. The *symmetric closure* of $\mathfrak{R}$ is the graph $S\mathfrak{R} = (V, S\mathcal{E})$, where $S\mathcal{E}$ denotes the set $\{(x,y) : (x,y) \in \mathcal{E} \text{ or } (y,x) \in \mathcal{E}\}$. Given any subset W of V, the relation *induced* on W by $\mathfrak{R}$ is $\mathfrak{R}(W) = \big(W, \mathcal{E} \cap (W \times W)\big)$. A relation on W is a *weak subrelation* of $\mathfrak{R}$, if all its edges are edges of $\mathfrak{R}(W)$. Given two sets V and V', to every $f : V' \to V$ is associated $\vec{f} : V' \times V' \to V \times V$, mapping each couple (x,y) to the couple $\big(f(x), f(y)\big)$. Given two relations $\mathfrak{R} = (V, \mathcal{E})$ and $\mathfrak{R}' = (V', \mathcal{E}')$, denote by $[\mathfrak{R}', \mathfrak{R}]$ the set of morphisms from $\mathfrak{R}'$ to $\mathfrak{R}$, ie. the set