Atomic compactness and reflexive graphs.

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Abstract

A first order structure $\mathfrak{M}$ with universe M is atomic compact if every system of atomic formulas with parameters in M is satisfiable in $\mathfrak{M}$ provided each of its finite subsystems is. The definition of atomic compactness goes back to Weglorz [Weg]. It extends to general first order structures the notion of equational compactness, first considered by I. Kaplansky and J. Løs for Abelian groups, and later by J. Mycielsky for general algebras (see [Myc]).

We present here miscellaneous results about atomic compactness of reflexive graphs.

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1 Preliminaries.

Let us recall some basic definitions and general facts about atomic compactness ; they appear in [Myc], [Weg] and [Myc,Ry-Na], see also [Weg] for details and examples.

Given a structure $\mathfrak{M} = (M; ...)$ with *universe* M over a first order language $\mathcal{L}$, an *atomic formula with parameters* in M is any formula $t_1 \equiv t_2$ (*equality formula*) or $r(t_1, \cdots t_n)$, where r is a relation symbol and $t_1, \cdots, t_n$ are terms of the language $\mathcal{L}_M$, obtained from $\mathcal{L}$ by the adjunction of the elements of M as new constant symbols ; *positive formulas* are built up from atomic ones using the connectives $\wedge$, $\vee$ and the quantifiers $\exists$, $\forall$; $(\exists, \wedge)$-*formulas* are obtained using only $\wedge$ and $\exists$. A *sentence* is a formula with no free variables.

A *system* of $\mathfrak{M}$ with *unknowns* in X is any set $\Phi(X)$ of formulas with parameters in M and with free variables in X ; the set of unknowns may be uncountably infinite. The system Φ is an *atomic*, resp. a *positive*, resp. a $(\exists, \wedge)$-*system* if its elements are respectivly atomic, positive or $(\exists, \wedge)$-formulas ; $\Phi^{\rm at}$ and Φ^{+} will denote respectively the set of atomic and positive elements of Φ. Given cardinals κ and λ, Φ is a κ-*system* (resp. a κ^{-}-*system*) if it has at most (resp. strictly less than) κ elements, it is a $[\lambda]$-*system* (resp. a $[\lambda^{-}]$-*system*) if it has at most (resp. strictly less than) λ unknowns. We shall freely combine these various prefixes.

Given a structure $\mathfrak{N} = (N; ...)$ over $\mathcal{L}_M$, a *solution* of Φ in $\mathfrak{N}$ is any family $(c_x : x \in X)$ of elements of N such that $\mathfrak{N}$ satisfies the set of sentences $\Phi[c_x : x \in X]$. The system Φ is *solvable* if it has a solution in $\mathfrak{M}$, it is *finitely solvable* if each of its finite subsets is solvable.

The structure $\mathfrak{M}$ is *atomic compact* if all its finitely solvable atomic systems are solvable. One similarly defines $\mathfrak{M}$ being compact with respect to any class of systems (e.g. $\mathfrak{M}$ is saturated if and only if it is $[1]$-card$(M)^{-}$-compact.)

The atomic compactness of $\mathfrak{M}$ can be checked by considering only "small" systems. Indeed, given any infinite cardinal κ greater or equal to the cardinality of $\mathcal{L}_M$, if $\mathfrak{M}$ is κ-atomic compact, then it is $[1]$-$(\exists, \wedge)$-compact, but then, any finitely solvable atomic system can be solved "one unknown after the other" (see [Myc,Ry-Na]).

Given an extension $\mathfrak{N} = (N; ...)$ of $\mathfrak{M} = (M; ...)$, recall that $\mathfrak{M}$ is a *retract* of $\mathfrak{N}$, if there exists a *retraction* from $\mathfrak{N}$ to $\mathfrak{M}$ (i.e. an $\mathcal{L}_M$-homomorphism, that is an $\mathcal{L}$-homomorphism fixing M pointwise) ; $\mathfrak{N}$ is an *elementary* (resp. a *pure*) *extension* of $\mathfrak{M}$, if $\mathfrak{M}$ and $\mathfrak{N}$ satisfy exactly the same sentences (resp. the same $(\exists, \wedge)$-sentences) with parameters in M.

Observe that if $\mathfrak{M}$ is a retract of $\mathfrak{N}$, then any positive system of $\mathfrak{M}$ with a solution in $\mathfrak{N}$ has a solution in $\mathfrak{M}$. Besides (by the compactness theorem) the finitely solvable systems of $\mathfrak{M}$ are exactly the systems which have a solution in an elementary extension of $\mathfrak{M}$ (they are the so called incomplete X-types of $\mathfrak{M}$ over M). But for each extension $\mathfrak{N}$ of $\mathfrak{M}$, the set of $\mathcal{L}_N$-sentences true in $\mathfrak{N}$ can be viewed as a system $\Phi(N - M)$ of $\mathfrak{M}$, with the property that any solution of $\Phi^{\rm at}$ defines a retraction of $\mathfrak{N}$ to $\mathfrak{M}$; moreover, Φ (resp. $\Phi^{\rm at}$) is finitely solvable precisely when $\mathfrak{N}$ is an elementary (resp. a pure) extension of $\mathfrak{M}$.

From these observations, together with the fact that homomorphisms onto "preserve" positive formulas (note incidentally that each retract of an atomic compact structure is