A Divide-and-Conquer Method for the Tridiagonal Generalized Eigenvalue Problem

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Abstract

A divide-and-conquer method is developed for solving the generalized eigenvalue problem $Ax = \lambda Bx$, where A, B are real symmetric tridiagonal and B is positive definite. It is a generalization of Cuppens method for the standard eigenvalue problem, where $B = I$, being based on rank one modifications. This method is complementing a method developed by Borges-Gragg using bordering modifications.

1 Introduction

In this note we discuss a numerical method for solving the tridiagonal eigenvalue problem

$$Ax = \lambda Bx, \qquad (1)$$

where A, B are real symmetric tridiagonal matrices and B is in addition positive definite.

In [3] Cuppen introduced a divide-and-conquer method for the case $B = I$. It is based on rank one modifications.

*Dedicated to W.B. Gragg at the occasion of his 60. birthday

Borges and Gragg, using modifications by bordering, developed later on in [1] a method for the general case (1). Independently Gu studied the bordering approach in [5] for the standard eigenvalue problem $B = I$.

Somewhat earlier, around 1988, the first author had sketched a generalization of Cuppens approach, namely using rank one modifications, to the case that B too is tridiagonal. In the Diplomarbeit [7] an algorithm based on these notes was developed and tested. As the numerical results seemed not to be very promising, they were never published outside [7]. Inspired by the publication of [1] the study of this topic was resumed in the Diplomarbeit [8]. By introducing a better solver for the secular equation numerically satisfying results were achieved. It turned out that further improvements were possible by using an updating for the eigenvectors developed by Gu-Eisenstat, see [6].

Encouraged by some people in the audience of a talk at the Householder meeting in Pontresina 1996, and in order to complete the picture we dare to publish these results belatedly.

2 The algorithm

In the sequel we develop an algorithm for (1). It consists of two phases, a *divide* or splitting phase, and a *conquer*

or construction phase.

Let

$$A-\lambda B = \begin{pmatrix} a_1 & \alpha_1 & & & 0 \\ \alpha_1 & a_2 & \alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & a_{n-1} & \alpha_{n-1} \\ 0 & & & \alpha_{n-1} & a_n \end{pmatrix} - \lambda \begin{pmatrix} b_1 & \beta_1 & & & 0 \\ \beta_1 & b_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & b_{n-1} & \beta_{n-1} \\ 0 & & & \beta_{n-1} & b_n \end{pmatrix} \tag{2}$$

In the divide phase we find a split index s, which is typically near $n/2$ and a vector v such that

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} - vv^T, \tag{3}$$

where B_1 is s -by - s. Here v is given by

$$v_s = \sqrt{(|\beta_s|)}, \quad v_{s+1} = -sgn(\beta_s)v_s, \quad v_j = 0 \quad else \tag{4}$$