EXPONENTIAL APPROXIMATIONS IN COMPLETELY REGULAR TOPOLOGICAL SPACES AND EXTENSIONS OF SANOV'S THEOREM

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Abstract. This paper is devoted to the well known transformations that preserve a large deviation principle (LDP), namely, the contraction principle with approximately continuous maps and the concepts of exponential equivalence and exponential approximations. We generalize these transformations to completely regular topological state spaces, give some examples and, as an illustration, reprove a generalization of Sanov's theorem, due to de Acosta [1]. Using partition-dependent couplings, we then extend this version of Sanov's theorem to triangular arrays and prove a LDP for the empirical measures of exchangeable sequences with a general measurable state space.

1. Introduction and statement of the results

The question when a large deviation principle (LDP) for a family of laws can be deduced from the LDP for another family was studied for the first time in [3]. There the probability measures are defined on a metric space. The concepts nowadays called "exponential equivalence" and "exponential approximations" were formalized more generally in the exposition [7, Chapter 4.2], where the probability measures are defined on a metric space, too. A direct consequence of this concept is the proof of a contraction principle for approximately continuous maps, where the maps are defined on a Hausdorff topological space and take values in a metric space. In this paper we will generalize the concepts of "exponential equivalence" and "exponential approximations" to a completely regular topological space $(Y, \mathcal{T})$, also called Tychonoff space, and we will generalize the contraction principle accordingly. The set $\mathcal{M}_1(S)$ of all probability measures on a general measurable space $(S, \mathcal{S})$, equipped with the so-called τ-topology of setwise convergence, is an example of such a completely regular topological space. Our general results allow us to reprove Sanov's theorem in this natural setting, extend it to triangular arrays and prove a LDP for the empirical measures of suitable exchangeable processes—without imposing restrictions on the measurable state space $(S, \mathcal{S})$. The basic tool for these extensions is the construction of couplings, which depend on finite $\mathcal{S}$-measurable partitions of S, to get exponentially good approximations.

Let us start with some topological considerations. A metric on a space Y can be regarded as providing a concept of nearness that is applicable throughout the space. When we want to consider approximations in more general topological spaces, we

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still need such a uniformly applicable concept of nearness. Two types of topological spaces seem to be appropriate for this purpose:

- A gauge space $(Y, \mathcal{T})$, which means that the topology $\mathcal{T}$ is generated by a family $\mathcal{D}$ of pseudometrics which is separating, that is, for each pair of points $x \neq y$ in Y there exists a pseudometric $d \in \mathcal{D}$ such that $d(x, y) \neq 0$.
- A topological space $(Y, \mathcal{T})$ with a separating uniform structure, which is compatible with the topology $\mathcal{T}$.

As is well-known [11, Chap. IX, Theorems 10.6 and 11.4], the above two types of topological spaces are the same, and they coincide with the completely regular ones. Note that every locally compact space and every Hausdorff topological vector space is completely regular (just construct a uniform structure for the latter).

To further justify the above topological setting, notice that in a regular topological space Y, the rate function associated with the LDP is unique and Varadhan's integral lemma is applicable. For Bryc's inverse Varadhan lemma, however, the complete regularity of the topological space is need (see [7, Chap. 4]).

For notational convenience, we will use the language of gauge spaces in the following. We will assume throughout this paper, that $(Y, \mathcal{T})$ is a gauge space and that the separating family $\mathcal{D}$ of pseudometrics generates its topology $\mathcal{T}$. Note that the collection of all balls

$$B(y, d, \delta) \equiv \{\, x \in Y \mid d(x, y) < \delta \,\}, \qquad y \in Y, d \in \mathcal{D}, \delta > 0, \tag{1.1}$$

is a subbasis of the topology $\mathcal{T}$. Let $\mathcal{D}'$ be the smallest family of pseudometrics on Y which contains $\mathcal{D}$ and is closed with respect to maxima, meaning that for every choice of $d_1, d_2 \in \mathcal{D}'$, the pseudometric defined by $d(x, y) \equiv \max\{d_1(x, y), d_2(x, y)\}$ for $x, y \in Y$ also belongs to $\mathcal{D}'$. The collection of all balls $B(y, d, \delta)$ with $y \in Y$, $d \in \mathcal{D}'$ and $\delta > 0$ is a basis of $\mathcal{T}$.

In this paper we take special care to mention any connection between the topology of a space and its σ-algebra, provided we need such a connection. In applications of our results (see Theorem 1.15, for instance), it is convenient, when the σ-algebra does not need to be the Borel σ-algebra. We do not complete our probability spaces.

With respect to the gauge space Y, we assume throughout that it is equipped with a σ-algebra $\mathcal{Y}$, which contains the above collection (1.1) of balls. Note that $\mathcal{Y}$ might be smaller than the Borel σ-algebra $\sigma(\mathcal{T})$. We denote by $\mathcal{M}_1(Y)$ the set of all probability measures on $(Y, \mathcal{Y})$.

Since in a gauge space the neighbourhood filterbase of a point $y \in Y$ is in general not countable, it is necessary to replace approximating sequences by approximating nets. For this purpose, let $(I, \preceq)$ denote a nonempty directed set, meaning that $\preceq$ is a reflexive and transitive relation such that for all $i, j \in I$ there exists a $k \in I$ satisfying $i \preceq k$ and $j \preceq k$. For a net $\{a_i\}_{i \in I} \subset \bar{\mathbb{R}}$ we define as usual $\limsup_{i \in I} a_i = \inf_{j \in I} \sup_{i \in I, i \succeq j} a_i$ and $\liminf_{i \in I} a_i = \sup_{j \in I} \inf_{i \in I, i \succeq j} a_i$.

Now we are ready to define the term "exponentially good approximation" and "exponential equivalence" in our context; illustrations are given in Section 2.

Definition 1.2. (a) A collection $\{\mu_{\varepsilon,i}\}_{\varepsilon>0, i \in I} \subset \mathcal{M}_1(Y)$ is called *$\mathcal{D}$-exponentially good approximation* of $\{\tilde{\mu}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}_1(Y)$, if for every $d \in \mathcal{D}'$, $\varepsilon > 0$ and $i \in I$ there exists a probability measure $\nu_{d,\varepsilon,i}$ on a σ-algebra $\mathcal{Y}_{d,\varepsilon,i}$ containing $\mathcal{Y}^{\otimes 2}$ such that