Kit Algebras

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Abstract

One of the main tools in representation theory of finite-dimensional algebras is the one-point extension technique. It allows to determine the representation type of an algebra whose quiver is obtained by iterated addition of sinks or sources. For large algebras, however, it seems more appropriate to glue together several parts at once. We introduce such a technique here. The resulting algebras are called kit algebras. We present a criterion when a kit algebra is tame.

In this paper we introduce a method to construct certain tame algebras by "glueing" smaller algebras together. These are called "kit algebras". This method extends the well-known one-point extension technique and uses essentially the Gelfand-type matrix problems:

Let Q be a quiver and I an admissible ideal of the path algebra kQ over an algebraically closed field k. We consider the algebra $A = kQ/I$. If s is a source in Q, then A is called a one-point extension of the algebra A_0, the latter being obtained from A by deleting the vertex s. The A-modules can be interpreted as the category of subspaces of A_0-modules [21]. This technique has been applied successfully to determine the representation type of an algebra, first in [10] for representation-finite hereditary algebras.

Recall that an algebra A is tame if for any dimension d there are finitely many 1-parameter families of A-modules which cover (up to isomorphism) almost all indecomposable d-dimensional A-modules. If the minimal number of such families in each dimension d is bounded by a polynomial in d, then A is called of polynomial growth, otherwise of non-polynomial growth.

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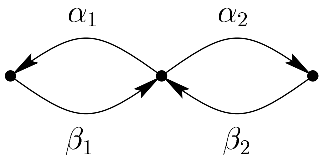
For non-polynomial growth tame algebras the one-point extension technique turns out to be inadequate and it seems more appropriate to glue together several algebras in one step instead of just adding one vertex at a time. But doing so, one has to replace the subspace category by a more general type of matrix problem.

This leads to the second ingredient, the Gelfand-type matrix problem. In 1970, I.M.Gelfand [13] reduced the study of Harish-Chandra-modules over $sl_2(\mathbb{C})$ to the classification of modules over the algebra given by the following quiver with relation $\alpha_1\beta_1 = \alpha_2\beta_2$:



Nazarova and Roiter [16] first showed that this algebra is tame (and of non-polynomial growth) using a class of "self-reproducing" matrix problems which we refer to as ones of "Gelfand-type". It is remarkable that this approach solved an entire class of matrix problems: The proof in [16] uses induction on the dimension of a module. The fact that the original problem is of non-polynomial growth leads in each induction step to a variety of new problems.

Crawley-Boevey [2] extended this result to a larger class of algebras, called "clannish algebras", and described their indecomposable representations. In his work, he used the functorial filtration method. The original approach of [16] was completed in [1] and [4], the latter including an abstract description of Gelfand-type matrix problems in terms of matrices over bimodules which turns out to be very useful for our approach.

We present in the first section a general method which reduces the study of modules over algebras to certain bimodule problems. In the second part we introduce the concept of a kit algebra and formulate the main theorem. The remaining two sections are devoted to the proof of this theorem and some applications.

We refer to [22] and [12] for general background on representations of finite-dimensional algebras.