UNIFORM RATES OF CONVERGENCE IN THE CLT FOR QUADRATIC FORMS IN MULTIDIMENSIONAL SPACES

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December 1996

Abstract. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random vectors taking values in a d-dimensional real linear space $\mathbb{R}^d$. Assume that $\mathbf{E}\, X = 0$ and that X is not concentrated in a proper subspace of $\mathbb{R}^d$. Let G denote a mean zero Gaussian random vector with the same covariance operator as that of X. We investigate the distributions of non-degenerate quadratic forms $\mathbb{Q}[S_N]$ of the normalized sums $S_N = N^{-1/2}\,(X_1 + \cdots + X_N)$ and show that

$$\Delta_N \overset{\text{def}}{=} \sup_x \Big|\, \mathbf{P}\,\big\{\, \mathbb{Q}[S_N] \le x \,\big\} - \mathbf{P}\,\big\{\, \mathbb{Q}[G] \le x \,\big\} \Big| = \mathcal{O}\big(\, N^{-1} \big),$$

provided that $d \ge 9$ and the fourth moment of X exists. The bound $\mathcal{O}\big(N^{-1}\big)$ is optimal and improves, e.g., the well-known bound $\mathcal{O}\big(N^{-d/(d+1)}\big)$ due to Esseen (1945). The result extends to the case of random vectors taking values in a Hilbert space. Furthermore, we provide explicit bounds for Δ_N and for the concentration function of the random variable $\mathbb{Q}[S_N]$.

1. Introduction

Let $\mathbb{R}^d$ denote the d-dimensional space of real vectors $x = (x_1, \ldots, x_d)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|x|^2 = \langle x, x \rangle = x_1^2 + \cdots + x_d^2$. Since our results are independent of the dimension (provided $d \ge 9$), it will be convenient to denote by $\mathbb{R}^\infty$ a real separable Hilbert space. Thus, $\mathbb{R}^\infty$ consists of all real sequences $x = (x_1, x_2, \ldots)$ such that $|x|^2 = x_1^2 + x_2^2 + \cdots < \infty$.

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random vectors taking values in $\mathbb{R}^d$. If $\mathbf{E}\, X = 0$ and $\mathbf{E}\, |X|^2 < \infty$, then the sums

$$S_N = N^{-1/2}\,(X_1 + \cdots + X_N)$$

1991 *Mathematics Subject Classification.* Primary 60F05; secondary 62E20.

Key words and phrases. Central Limit Theorem, concentration inequalities, convergence rates, Berry–Esseen bounds, Edgeworth expansions, multidimensional spaces, Hilbert spaces, quadratic forms, ellipsoids, hyperboloids, lattice point problem.

[1]Research supported by the SFB 343 in Bielefeld.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

converge weakly to a mean zero Gaussian random vector, say G, such that its covariance operator $\mathbb{C} = \operatorname{cov} G : \mathbb{R}^d \to \mathbb{R}^d$ is equal to $\operatorname{cov} X$.

For a linear symmetric and bounded operator $\mathbb{Q} : x \mapsto \mathbb{Q}x$ mapping $\mathbb{R}^d$ into $\mathbb{R}^d$, define the quadratic form $\mathbb{Q}[x] = \langle \mathbb{Q}x, x \rangle$. We call $\mathbb{Q}$ non-degenerate if $\ker \mathbb{Q} = \{ 0 \}$, or equivalently, if $\mathbb{Q}$ is injective. If $d < \infty$, non-degeneracy means that $\mathbb{Q}$ is invertible.

Notice that the distribution of the quadratic form $\mathbb{Q}[G]$ may be represented up to a shift as the distribution of a finite (resp. eventually infinite) weighted sum of squares of i.i.d. standard Gaussian variables, for $d < \infty$ (resp., $d = \infty$).

Write

$$\sigma^2 = \beta_2, \qquad \beta_q = \mathbf{E}\,|X|^q, \qquad \text{for } q \geq 0.$$

Theorem 1.1. *Let* $\mathbf{E}\,X = 0$. *Assume that* $\mathbb{Q}$ *and* $\mathbb{C}$ *are non-degenerate and that* $d \geq 9$ *or* $d = \infty$. *Then*

$$\sup_x \Big| \mathbf{P}\left\{ \mathbb{Q}[S_N] \leq x \right\} - \mathbf{P}\left\{ \mathbb{Q}[G] \leq x \right\} \Big| \leq c(\mathbb{Q}, \mathbb{C})\, \beta_4 / N.$$

The constant $c(\mathbb{Q}, \mathbb{C})$ *in this bound depends on* $\mathbb{Q}$ *and* $\mathbb{C}$ *only.*

Remark. A rather straightforward inspection of the proofs shows that Theorem 1.1 holds for $d = 8$ with $\mathcal{O}(N^{-1} \log^\delta N)$ instead of $\mathcal{O}(N^{-1})$, for some $\delta > 0$. It is likely that the results of the paper remain valid for $d \geq 5$. We need the assumption $d \geq 9$ for estimation of integrals over the characteristic function of the quadratic form for Fourier frequencies $N^{3/5} \ll |t| \ll N$.

An earlier version of results of this paper were published in Bentkus and Götze (1995b).

The bound of Theorem 1.1 is optimal since the distribution function of $|S_N|^2$ (for bounded $X \in \mathbb{R}^d$) may have jumps of order $\mathcal{O}\big(N^{-1}\big)$, for all $1 \leq d \leq \infty$. See, for example, Bentkus and Götze (1996). In that paper a similar bound $\mathcal{O}(N^{-1})$ as in Theorem 1.1 is proved even for $d \geq 5$ assuming that $\mathbb{Q}$ is diagonal on the subspace spanned by five coordinates of X, which have to be stochastically independent and independent of other coordinates. Both results are based on discretization (i.e., a reduction to lattice valued random vectors) and symmetrization techniques. The independence assumption in Bentkus and Götze (1996) allowed to apply an adaption of the Hardy–Littlewood circle method. For the general case described in Theorem 1.1, we had to develop a new tool—a multiplicative inequality for characteristic functions.

Theorem 1.1 and the method of its proof are related to the well known lattice point problem for conic sections in number theory. Assume that $\mathbb{R}^d$ is finite dimensional and that $\langle \mathbb{Q}x, x \rangle > 0$, for $x \neq 0$. Write $\operatorname{vol} E_s$ for the volume of the ellipsoid

$$E_s = \left\{ x \in \mathbb{R}^d : \ \mathbb{Q}[x] \leq s \right\}, \qquad \text{for} \quad s \geq 0.$$

Let $\operatorname{vol}_{\mathbb{Z}} E_s$ be the number of points in $E_s \cap \mathbb{Z}^d$, where $\mathbb{Z}^d \subset \mathbb{R}^d$ is the standard lattice of points with integer coordinates.

The following result corresponds to Theorem 1.1.