A projection property for buildings

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An object X of a category is said to have the *projection property* if the only idempotent morphisms $f : X \times X \to X$ are the projections. Here a morphism $f : X \times X \to X$ is called idempotent if $f \circ \Delta = id$ for the diagonal map $\Delta : X \to X \times X$.

There are two motivations for studying the question whether X has the projection property. Firstly, Arrow's "dictator theorem" states that the only maps of a product X^A to X with certain properties are the projections (see [A], [PRS]). Secondly, the projection property is closely related to the fixed point property (see [C]). For a detailed discussion of the projection property, its background and connections with other properties see the paper [P] in the same volume.

In this paper we prove that an irreducible building of spherical type and of rank at least 2 has the projection property. Actually, the theorem is more general. For a precise statement of the hypotheses see below. By contrast, every reducible building and every building of rank one does not have the projection property. We also give a counterexample concerning the finiteness assumption of the theorem.

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§ 1 Statement of the theorem

Let $\mathcal{C}$ be a building, which we think of as the set of its chambers endowed with a W –distance $\delta : \mathcal{C} \times \mathcal{C} \to W$ where W is the Weyl group of the building, see e.g. [Sch 3.3.12]. So W is a Coxeter group endowed with a set $\{s_i | i \in I\}$ of generators subject to the defining relations

$$s_i^2 = (s_i s_j)^{m_{ij}} = 1 \quad .$$

Here $(m_{ij})_{i,j\in I}$ is a Coxeter matrix, i.e. we have

$$\begin{aligned} m_{ii} &= 2 \\ 2 \le m_{ij} &= m_{ji} \le \infty \quad . \end{aligned}$$

The corresponding *Coxeter graph* has the set I for its set of vertices and the vertices i, j are joined by an $m_{ij} - 2$ –fold edge. So i and j are not joined by an edge if $m_{ij} = 2$. We say I is connected, if the Coxeter graph is connected. The number $r = \#I$ is called the *rank* of $\mathcal{C}$ or of W .

Two chambers x, y are called *i–adjacent* if $\delta(x,y) \in \{1, s_i\}$. We write $x \sim_i y$. For $J \subset I$ define the equivalence relation $\sim_J$ by $x \sim_J y$ iff $\delta(x,y)$ is in the subgroup of W generated by $\{s_j, j \in J\}$.

The building $\mathcal{C}$ is said to have the *projection property* if the only maps $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the following properties are the two projections $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$:

$$\begin{array}{ll} (1) & f(x,x) = x \\ (2a) & f(x,y) \sim_i f(x',y) \ \text{ if } \ x \sim_i x' \\ (2b) & f(x,y) \sim_i f(x,y') \ \text{ if } \ y \sim_i y' \quad . \end{array}$$

Let us call a map with these properties an idempotent morphism. This fits under the language of categories as follows. Regard $\mathcal{C}$ as a set with a family of equivalence relations $\sim_i$, indexed by elements i of I . In the category of sets with a family of equivalence relations $\sim_i, i \in I$, the set $\mathcal{C} \times \mathcal{C}$ together with the equivalence relations $(x,y) \sim_i (x',y')$ iff $x \sim_i x'$ and $y \sim_i y'$, is a product $\mathcal{C} \times \mathcal{C}$. Then $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a morphism in this category iff f satisfies $2a$ and $2b$, and f satisfies (1) iff $f \circ \Delta = id$ for the diagonal morphism $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

Theorem
Let $\mathcal{C}$ be a building. Suppose its Coxeter diagram is connected, its rank is at least two and all the Coxeter numbers are finite. Then $\mathcal{C}$ has the projection property.

In particular, every spherical building of rank at least 2 with connected diagram has the projection property. An example of such a building is the set of maximal flags in a vector space of dimension $n \ge 3$. Another example – actually the Coxeter complex of the preceding example – is the set of all maximal chains in the poset of all subsets of a finite set of cardinality at least 3. This special case proved in [PRS] was actually