On the structure of cellular algebras

by

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1 Introduction

Recently, Graham and Lehrer [17, 16] have introduced a new class of algebras, which they called cellular algebras. These are defined by the existence of a certain basis with very special properties motivated by the Kazhdan–Lusztig basis of Hecke algebras. Graham and Lehrer have shown that various important classes of algebras are cellular, in particular, Ariki–Koike Hecke algebras, and Brauer (centralizer) algebras.

The aim of this paper is to study the structure of cellular algebras, in particular in the case of the ground ring being a field (this is assumed in sections four to seven). We start by shortly describing the two classes of motivating examples just mentioned. In section three we review some of the results of Graham and Lehrer and give an equivalent form of their definition, and some examples.

In section four we investigate the cell ideals (which are the building blocks of cellular algebras). In particular we show that there are two different sorts of such ideals, one being familiar from (and thus providing a close connection to) the theory of quasi–hereditary algebras. In section five we show that the involution occuring in the definition of cellular algebras must fix isomorphism classes of simple modules. This is a strong restriction, as we illustrate by the example of Brauer tree algebras.

In section six we collect some homological properties for later use. In section seven we give an inductive construction of cellular algebras. This

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provides us with a method to produce many examples, which in particular show that there are no restrictions on the cell ideals e.g. with respect to endomorphism rings - which is a strong contrast to the situation for quasi–hereditary algebras. We also give a bound on the Loewy length of a cellular algebra in terms of the number of cell ideals. Moreover we determine the global dimension of certain cellular algebras.

This paper is the first in a series of three papers. In the subsequent paper [25] we define integral cellular algebras. This is a smaller class of algebras, still containing the examples of [17], but with much stronger properties than cellular algebras in general. The paper [26] contains a new characterization of cellular algebras which gives another inductive construction of cellular algebras and which leads to a generalization of Hochschild cohomology.

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2 Motivating examples

In representation theory of Lie algebras and algebraic groups several classes of finite dimensional associative algebras play important roles. Most of them are defined by generators and relations. The motivation for studying cellular algebras or related classes of algebras (like quasi–hereditary algebras) is to understand these special classes of examples by putting them into an axiomatic framework and thus to reveal hidden structure. In this section we give a list of those examples which at present seem to be the most interesting ones. Details are given only for those examples which are not already covered by the literature on quasi–hereditary algebras.

2.1 Ariki Koike Hecke algebras

Hecke algebras of type A are well–known deformations of the group algebra of the symmetric group (for a survey see e.g. [10]). They have been studied and used intensively in many areas of mathematics. For the other classical Coxeter groups (type B, C, and so on), Hecke algebras have been defined as well. Recently, several attempts have been made to define Hecke algebras in a more general context, for socalled complex reflection groups. The so–called cyclotomic Hecke algebras are closely related to a system of results and conjectures which has been built around Broué's conjecture (on derived equivalences for blocks with abelian defect group). For more information