Orthogonal Gelfand-Zetlin Algebras, I

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Abstract

We define a class of assosiative algebras those are similar to enveloping algebra of $gl(n, \mathbb{C})$. We construct a family of simple Verma and generalized Verma modules over such algebras. We also investigate two simpliest classes of such algebras in detailes. For the first class we construct all finite-dimensional modules and investigate an $\mathcal{O}$-category and for the second one we classify all weight simple modules.

1 Introduction

During last time several examples of algebras which are similar to the enveloping algebras of simple finite-dimensional complex Lie algebras appear as good as the representation theory of such algebras (see for example [1, 10, 12]). Till now all "good" examples were obtained by generalization of an algebra $sl(2, \mathbb{C})$ as it done in [1, 12]. Careful study of such algebras made it possible to obtain good results even for more difficult algebras ([2]) those are tenzor products of some copies of algebras similar to $sl(2, \mathbb{C})$. But there were no constructions which allows one to obtait some non-trivial generalization of Lie algebras of higher rank.

In the subsequent article we propose a construction of a new family of assosiative algebras that arose from the outstanding paper [8]. Using the formulas that describes an action of generating elements of $gl(n, \mathbb{C})$ on finite-dimensional modules we construct a new family of assosiative algebras that we call Orthogonal Gelfand-Zetlin algebras. It seems that the representation theory of such algebras has common features with the representation theory of $gl(n, \mathbb{C})$.

We show that the family of Orthogonal Gelfand-Zetlin algebras contains $gl(n, \mathbb{C})$, polinomial algebras and extended Heisenberg algebra. Note that the definition of our algebras is not too comfortable. We define Orthogonal Gelfand-Zetlin algebra as an operator algebra generated by some special operators in infinite-dimensional space. So it is not too simple to work with such algebras untill there were no analogue of PBW-theorem.

The positive side of our definition is that any Orthogonal Gelfand-Zetlin algebra appears both with a huge family of modules obtained by specialization of the formulas that defines the algebra itself. This allows us to investigate weight modules over our algebras in two simpliest cases: the first one is an analogue to the case of $sl(2, \mathbb{C})$ and the second one is the case of algebras for which GZ-formulas (formulas 1,2) has polinomial coefficients.

In the subsequent papers we will disscus some questions conserning the structure of Verma and generalized Verma modules over arbitrary Orthogonal Gelfand-Zetlin algebras, will try to develop the structure theory and describe all finite-dimensional simple modules in terms of highest weight.

Let us briefly describe the structure of the paper. In section 2 we collect all basic preliminaries. In section 3 we define Orthogonal Gelfand-Zetlin algebras and describe their basic properties. In section 4 we prove that $gl(n, \mathbb{C})$ is Orthogonal Gelfand-Zetlin algebra (theorem 1). In section 5 we obtain some identities in $\mathcal{U}$ and find a natural Harish-Chandra subalgebra in $\mathcal{U}$ (corollary 1). In section 6 we give some examples of Orthogonal Gelfand-Zetlin algebras. In sections 7-8 we construct two families of simple $\mathcal{U}$-modules which are analogues to Verma and generalized Verma modules over Lie algebra. In sections 9-11 we investigate first special class of algebras those are rather similar to $sl(2, \mathbb{C})$. In particular we describe for them all simple finite-dimensional modules (theorem 4) and prove the BGG-duality in the corresponding category $\mathcal{O}$ (theorem 5). In the last section 12 we classify weight simple modules for another class of Orthogonal Gelfand-Zetlin algebras (theorem 6).

2 Preliminaries

Let $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{Z}$ denotes a ring of integers, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{Z}_+$ denotes the set of all non-negative integers.

Throughout this paper field means field of zero characteristics and assosiative algebra means assosiative algebra with identity.

Fix $n \in \mathbb{N}$ and $r = (r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$. For a fixed field $\mathbb{F}$ and $k = |r| = \sum_{i=1}^{n} r_i$ consider a vectorspace $\mathcal{L} = \mathcal{L}(\mathbb{F}, r) = \mathbb{F}^k$. We will call the elements from $\mathcal{L}$ tableaux and will consider them as double indexed families

$$[\,l\,] = \{l_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, r_i\}.$$

For $[\,l\,] \in \mathcal{L}$ and $i \in \{1, 2, \ldots, n\}$ we will denote by $[\,l\,]_i = \{l_{ij} \mid j = 1, \ldots, r_i\}$ the i-th row of $[\,l\,]$. An element r will be called signature of $[\,l\,]$. We also set $\mathrm{rank}([\,l\,]) = n - 1$.

We will denote by δ^{ij}, $1 \le i \le n$, $1 \le j \le r_i$ the Kronecker tableaux, i.e. $\delta^{ij}_{ij} = 1$ and other coordinates are zero.

Denote by $\mathcal{L}_0$ the subset $\mathcal{L}$ that consists of all $[\,l\,]$ satisfying following conditions:

1. $l_{nj} = 0$, $j = 1, \ldots, r_n$;
2. $l_{ij} \in \mathbb{Z}$, $1 \le i \le n-1$, $1 \le j \le r_i$.

3 Orthogonal Gelfand-Zetlin algebras

Fix some $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$. Set $k = |r|$. Consider a field Λ of rational functions in k variables λ_{ij}, $1 \le i \le n$, $1 \le j \le r_i$. Let $[\,\mathfrak{l}\,] \in \mathcal{L}(\Lambda, r)$ be the tableaux defined by $\mathfrak{l}_{ij} = \lambda_{ij}$, $1 \le i \le n$, $1 \le j \le r_i$.